FUZZY MULTI-CRITERIA DECISION MAKING

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FUZZY MULTI-CRITERIA DECISION MAKING

Theory and Applications with Recent Developments

Edited By

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PREFACE

Multiple criteria decision making (MCDM) is a modeling and methodological tool for dealing with complex engineering problems. Decision makers face many problems with incomplete and vague information in MCDM problems since the characteristics of these problems often require this kind of information. Fuzzy set approaches are suitable to use when the modeling of human knowledge is necessary and when human evaluations are needed. Fuzzy set theory is recognized as an important problem modeling and solution technique. Fuzzy set theory has been studied extensively over the past 40 years. Most of the early interest in fuzzy set theory pertained to representing uncertainty in human cognitive processes. Fuzzy set theory is now applied to problems in engineering, business, medical and related health sciences, and the natural sciences. Over the years there have been successful applications and implementations of fuzzy set theory in MCDM. MCDM is one of the branches in which fuzzy set theory found a wide application area. Many curriculums of undergraduate and graduate programs include many courses teaching how to use fuzzy sets when you face incomplete and vague information. One of these courses is fuzzy MCDM and its applications.

This book presents examples of applications of fuzzy sets in MCDM. It contains 22 original research and application chapters from different perspectives; and covers different areas of fuzzy MCDM. The book contains chapters on the two major areas of MCDM to which fuzzy set theory contributes. These areas are fuzzy multiple-attribute decision making (MADM) and fuzzy multiple-objective decision making (MODM). MADM approaches can be viewed as alternative methods for combining the information in a problem's decision matrix together with additional information from the decision maker to determine a final ranking, screening, or selection from among the alternatives. MODM is a powerful tool to assist in the process of searching for decisions that best satisfy a multitude of conflicting objectives.

The classification, review and analysis of fuzzy multi-criteria decision-making methods are summarized in the first two chapters. While the first chapter classifies the multi-criteria methods in a general sense, the second chapter focuses on intelligent fuzzy multi-criteria decision making.

The rest of the book is divided into two main parts. The first part includes chapters on frequently used MADM techniques under fuzziness, e.g., fuzzy Analytic Hierarchy Process (AHP), fuzzy TOPSIS, fuzzy outranking methods, fuzzy weighting methods, and a few application chapters of these techniques. The third chapter includes the most frequently used fuzzy AHP methods and their numerical and didactic examples. The fourth chapter shows how a fuzzy AHP method can be jointly used with another technique. The fifth chapter summarizes fuzzy outranking methods, which dichotomize preferred alternatives and nonpreferred ones by establishing outranking relationships. The sixth chapter presents another commonly used multi-attribute method, fuzzy TOPSIS and its application to selection among industrial robotic systems. The seventh chapter includes many fuzzy scoring methods and their applications. The rest of this part includes the other most frequently used fuzzy MADM techniques in the literature: fuzzy information axiom approach, intelligent fuzzy MADM approaches, gray-related analysis, and neuro-fuzzy approximation.

The second part of the book includes chapters on MODM techniques under fuzziness, e.g., fuzzy multi-objective linear programming, quasi-concave and non-concave fuzzy multi-objective programming, interactive fuzzy stochastic linear programming, fuzzy multi-objective integer goal programming, gray fuzzy multi-objective optimization, fuzzy multi-objective geometric programming and some applications of these techniques. These methods are the most frequently used MODM techniques in the fuzzy literature.

The presented methods in this book have been prepared by the authors who are the developers of these techniques. I hope that this book will provide a useful resource of ideas, techniques, and methods for additional research on the applications of fuzzy sets in MCDM. I am grateful to the referees whose valuable and highly appreciated works contributed to select the high quality of chapters published in this book. I am also grateful to my research assistant, Dr. Ihsan Kaya, for his invaluable effort to edit this book.

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May 2008

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MULTI-CRITERIA DECISION MAKING METHODS AND FUZZY SETS

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Abstract: Multi-criteria decision making (MCDM) is one of the well-known topics of decision making. Fuzzy logic provides a useful way to approach a MCDM problem. Very often in MCDM problems, data are imprecise and fuzzy. In a real-world decision situation, the application of the classic MCDM method may face serious practical constraints, because of the criteria containing imprecision or vagueness inherent in the information. For these cases, fuzzy multi-attribute decision making (MADM) and fuzzy multi-objective decision making (MODM) methods have been developed. In this chapter, crisp MADM and MODM methods are first summarized briefly and then the diffusion of the fuzzy set theory into these methods is explained. Some examples of recently published papers on fuzzy MADM and MODM are given.

Key words: Multi-criteria, multi-attribute, multi-objective, decision making, fuzzy sets

1. INTRODUCTION

In the literature, there are two basic approaches to multiple criteria decision making (MCDM) problems: multiple attribute decision making (MADM) and multiple objective decision making (MODM). MADM problems are distinguished from MODM problems, which involve the design of a “best” alternative by considering the tradeoffs within a set of interacting design constraints. MADM refers to making selections among some courses of action in the presence of multiple, usually conflicting, attributes. In MODM problems, the number of alternatives is effectively

C. Kahraman (ed.), Fuzzy Multi-Criteria Decision Making.

infinite, and the tradeoffs among design criteria are typically described by continuous functions.

MADM is the most well-known branch of decision making. It is a branch of a general class of operations research models that deal with decision problems under the presence of a number of decision criteria. The MADM approach requires that the choice (selection) be made among decision alternatives described by their attributes. MADM problems are assumed to have a predetermined, limited number of decision alternatives. Solving a MADM problem involves sorting and ranking.

MADM approaches can be viewed as alternative methods for combining the information in a problem's decision matrix together with additional information from the decision maker to determine a final ranking, screening, or selection from among the alternatives. Besides the information contained in the decision matrix, all but the simplest MADM techniques require additional information from the decision maker to arrive at a final ranking, screening, or selection.

In the MODM approach, contrary to the MADM approach, the decision alternatives are not given. Instead, MODM provides a mathematical framework for designing a set of decision alternatives. Each alternative, once identified, is judged by how close it satisfies an objective or multiple objectives. In the MODM approach, the number of potential decision alternatives may be large. Solving a MODM problem involves selection.

It has been widely recognized that most decisions made in the real world take place in an environment in which the goals and constraints, because of their complexity, are not known precisely, and thus, the problem cannot be exactly defined or precisely represented in a crisp value (Bellman and Zadeh, 1970). To deal with the kind of qualitative, imprecise information or even ill-structured decision problems, Zadeh (1965) suggested employing the fuzzy set theory as a modeling tool for complex systems that can be controlled by humans but are hard to define exactly.

Fuzzy logic is a branch of mathematics that allows a computer to model the real world the same way that people do. It provides a simple way to reason with vague, ambiguous, and imprecise input or knowledge. In Boolean logic, every statement is true or false; i.e., it has a truth value 1 or 0. Boolean sets impose rigid membership requirements. In contrast, fuzzy sets have more flexible membership requirements that allow for partial membership in a set. Everything is a matter of degree, and exact reasoning is viewed as a limiting case of approximate reasoning. Hence, Boolean logic is a subset of Fuzzy logic. Human beings are involved in the decision analysis since decision making should take into account human subjectivity, rather than employing only objective probability measures. This makes fuzzy decision making necessary.

This chapter aims at classifying MADM and MODM methods and at explaining how the fuzzy sets have diffused into the MCDM methods.

2. MULTI-ATTRIBUTE DECISION MAKING: A CLASSIFICATION OF METHODS

MADM methods can be classified as to whether if they are non-compensatory or compensatory. The decision maker may be of the view that high performance relative to one attribute can at least partially compensate for low performance relative to another attribute, particularly if an initial screening analysis has eliminated alternatives that fail to meet any minimum performance requirements. Methods that incorporate tradeoffs between high and low performance into the analysis are termed "compensatory." Those methods that do not are termed "noncompensatory."

In their book, Hwang and Yoon (1981) give 14 MADM methods. These methods are explained briefly below. Additionally five more methods are listed below.

2.1 Dominance

An alternative is "dominated" if another alternative outperforms it with respect to at least one attribute and performs equally with respect to the remainder of attributes. With the dominance method, alternatives are screened such that all dominated alternatives are discarded. The screening power of this method tends to decrease as the number of independent attributes becomes larger.

2.2 Maximin

The principle underlying the maximin method is that "a chain is only as strong as its weakest link." Effectively, the method gives each alternative a score equal to the strength of its weakest link, where the "links" are the attributes. Thus, it requires that performance with respect to all attributes be measured in commensurate units (very rare for MADM problems) or else be normalized prior to performing the method.

2.3 Maximax

The viewpoint underlying the maximax method is one that assigns total importance to the attribute with respect to which each alternative performs best. Extending the "chain" analogy used in describing the maximin method, maximax performs as if one was comparing alternative chains in search of the best link. The score of each chain (alternative) is equal to the performance of its strongest link (attribute). Like the maximin method, maximax requires that all attributes be commensurate or else pre-normalized.

2.4 Conjunctive (Satisficing)

The conjunctive method is purely a screening method. The requirement embodied by the conjunctive screening approach is that to be acceptable, an alternative must exceed given performance thresholds for all attributes. The attributes (and thus the thresholds) need not be measured in commensurate units.

2.5 Disjunctive

The disjunctive method is also purely a screening method. It is the complement of the conjunctive method, substituting "or" in place of "and." That is, to pass the disjunctive screening test, an alternative must exceed the given performance threshold for at least one attribute. Like the conjunctive method, the disjunctive method does not require attributes to be measured in commensurate units.

2.6 Lexicographic

The best-known application of the lexicographic method is, as its name implies, alphabetical ordering such as is found in dictionaries. Using this method, attributes are rank-ordered in terms of importance. The alternative with the best performance on the most important attribute is chosen. If there are ties with respect to this attribute, the next most important attribute is considered, and so on. Note two important ways in which MADM problems typically differ from alphabetizing dictionary words. First, there are many fewer alternatives in a MADM problem than words in the dictionary. Second, when the decision matrix contains quantitative attribute values, there are effectively an infinite number [rather than 26 (i.e., A-Z)] of possible scores with a correspondingly lower probability of ties.

2.7 Lexicographic Semi-Order

This is a slight variation on the lexicographic method, where "near-ties" are allowed to count as ties without any penalty to the alternative, which scores slightly lower within the tolerance ("tie") window. Counting near-ties as ties makes the lexicographic method less of a "knife-edged" ranking method and more appropriate for MADM problems with quantitative data in the decision matrix. However, the method can lead to intransitive results, wherein A is preferred to B, B is preferred to C, but C is preferred to A.

2.8 Elimination by Aspects

This method is a formalization of the well-known heuristic, "process of elimination." Like the lexicographic method, this evaluation proceeds one attribute at a time, starting with attributes determined to be most important. Then, like the conjunctive method, alternatives not exceeding minimum performance requirements—with respect to the single attribute of interest, in this case—are eliminated. The process generally proceeds until one alternative remains, although adjustment of the performance threshold may be required in some cases to achieve a unique solution.

2.9 Linear Assignment Method

This method requires, in addition to the decision matrix data, cardinal importance weights for each attribute and rankings of the alternatives with respect to each attribute. These information requirements are intermediate between those of the eight methods described previously, and the five methods that follow, in that they require ordinal (but not cardinal) preference rankings of the alternatives with respect to each attribute. The primary use of the additional information is to enable compensatory rather than noncompensatory analysis, that is, allowing good performance on one attribute to compensate for low performance on another.

Note at this point that quantitative attribute values (data in the decision matrix) do not constitute cardinal preference rankings. Attribute values are generally noncommensurate across attributes, preference is not necessarily linearly increasing with attribute values, and preference for attribute values of zero is not generally zero. However, as long as the decision maker can specify an ordinal correspondence between attribute values and preference, such as “more is better” or “less is better” for each attribute, then the ordinal alternative rankings with respect to each attribute that are needed by the linear assignment method are specified uniquely. Thus, the evaluation/performance rankings required by the linear assignment method are easier to derive than the evaluation/performance ratings required by the five methods that follow. The cost of using ordinal rankings rather than cardinal ratings is that the method is only “semi-compensatory,” in that incremental changes in the performance of an alternative will not enter into the analysis unless the changes are large enough to alter the rank order of the alternatives.

2.10 Additive Weighting

The score of an alternative is equal to the weighted sum of its cardinal evaluation/preference ratings, where the weights are the importance weights associated with each attribute. The resulting cardinal scores for each alternative can be used to rank, screen, or choose an alternative. The analytical hierarchy process (AHP) is a particular approach to the additive weighting method.

2.11 Weighted Product

The weighted product is similar to the additive weighting method. However, instead of calculating “sub-scores” by multiplying performance scores times attribute importances, performance scores are raised to the power of the attribute importance weight. Then, rather than summing the resulting subscores across attributes to yield the total score for the alternative, the product of the scores yields the final alternative scores. The weighted product method tends to penalize poor performance on one attribute more heavily than does the additive weighting method.

2.12 Nontraditional Capital Investment Criteria

This method entails pairwise comparisons of the performance gains (over a baseline alternative) among attributes, for a given alternative. One attribute must be measured in monetary units. These comparisons are combined to estimate the (monetary) value attributed to each performance gain, and these values are summed to yield the overall implied value of each alternative. These implied values can be used to select an alternative, to rank alternatives, or presumably to screen alternatives as well.

2.13 TOPSIS (Technique for Order Preference by Similarity to Ideal Solution)

The principle behind TOPSIS is simple: The chosen alternative should be as close to the ideal solution as possible and as far from the negative-ideal solution as possible. The ideal solution is formed as a composite of the best performance values exhibited (in the decision matrix) by any alternative for each attribute. The negative-ideal solution is the composite of the worst performance values. Proximity to each of these performance poles is measured in the Euclidean sense (e.g., square root of the sum of the squared distances along each axis in the “attribute space”), with optional weighting of each attribute.

2.14 Distance from Target

This method and its results are also straightforward to describe graphically. First, target values for each attribute are chosen, which need not be exhibited by any available alternative. Then, the alternative with the shortest distance (again in the Euclidean sense) to this target point in “attribute space” is selected. Again, weighting of attributes is possible. Distance scores can be used to screen, rank, or select a preferred alternative.

2.15 Analytic Hierarchy Process (AHP)

The analytical hierarchy process was developed primarily by Saaty (1980). AHP is a type of additive weighting method. It has been widely reviewed and applied in the literature, and its use is supported by several commercially available, user-friendly software packages. Decision makers often find it difficult to accurately determine cardinal importance weights for a set of attributes simultaneously. As the number of attributes increases, better results are obtained when the problem is converted to one of making a series of pairwise comparisons. AHP formalizes the conversion of the attribute weighting problem into the more tractable problem of making a series of pairwise comparisons among competing attributes. AHP summarizes the results of pairwise comparisons in a “matrix of pairwise comparisons.” For each pair of attributes, the decision maker specifies a judgment about "how much more important one attribute is than the other."

Each pairwise comparison requires the decision maker to provide an answer to the question: "Attribute *A* is how much more important than Attribute *B*, relative to the overall objective?"

2.16 Outranking Methods (ELECTRE, PROMETHEE, ORESTE)

The basic concept of the ELECTRE (ELimination Et Choix Traduisant la Réalité or Elimination and Choice Translating Reality) method is how to deal with outranking relation by using pairwise comparisons among alternatives under each criteria separately. The outranking relationship of two alternatives, denoted as $A_i \rightarrow A_j$, describes that even though two alternatives *i* and *j* do not dominate each other mathematically, the decision maker accepts the risk of regarding A_i as almost surely better than A_j. An alternative is dominated if another alternative outranks it at least in one criterion and equals it in the remaining criteria. The ELECTRE method consists of a pairwise comparison of alternatives based on the degree to which evaluation of the alternatives and preference weight confirms or contradicts the pairwise dominance relationship between the alternatives. The decision maker may declare that she/he has a strong, weak, or indifferent preference or may even be unable to express his or her preference between two compared alternatives. The other two members of outranking methods are PROMETHEE and ORESTE.

2.17 Multiple Attribute Utility Models

Utility theory describes the selection of a satisfactory solution as the maximization of satisfaction derived from its selection. The best alternative is the one that maximizes utility for the decision maker's stated preference structure. Utility models are of two types additive and multiplicative utility models. The main steps in using a multi-attribute utility model can be counted as 1) determination of utility functions for individual attributes, 2) determination of weighting or scaling factors, 3) determination of the type of utility model, 4) the measurement of the utility values for each alternative with respect to the considered attributes, and 5) the selection of the best alternative.

2.18 Analytic Network Process

In some practical decision problems, it seems to be the case where the local weights of criteria are different for each alternative. AHP has a difficulty in treating in such a case since AHP uses the same local weights of criteria for each alternative. To overcome this difficulty, Saaty (1996) proposed the analytic network process (ANP). ANP permits the use of different weights of criteria for alternatives.

2.19 Data Envelopment Analysis

Data envelopment analysis (DEA) is a nonparametric method of measuring the efficiency of a decision making unit such as a firm or a public-sector agency, which was first introduced into the operations research literature by Charnes et al. (1978). DEA is a relative, technical efficiency measurement tool, which uses operations research techniques to automatically calculate the weights assigned to the inputs and outputs of the production units being assessed. The actual input/output data values are then multiplied with the calculated weights to determine the efficiency scores. DEA is a nonparametric multiple criteria method; no production, cost, or profit function is estimated from the data.

2.20 Multi-Attribute Fuzzy Integrals

When mutual preferential independence among criteria can be assumed, consider that the utility function is additive and takes the form of a weighted sum. The assumption of mutual preferential independence among criteria is, however, rarely verified in practice. To be able to take interaction phenomena among criteria into account, it has been proposed to substitute a monotone set function on attributes set N called the fuzzy measure to the weight vector involved in the calculation of weighted sums. Such an approach can be regarded as taking into account not only the importance of each criterion but also the importance of each subset of criteria. Choquet integral is a natural extension of the weighted arithmetic mean (Grabisch, 1992; Sugeno, 1974).

3. MULTI-OBJECTIVE DECISION MAKING: A CLASSIFICATION OF METHODS

In multiple objective decision making, application functions are established to measure the degree of fulfillment of the decision maker's requirements (achievement of goals, nearness to an ideal point, satisfaction, etc.) on the objective functions and are extensively used in the process of finding "good compromise" solutions. MODM methodologies can be categorized in a variety of ways, such as the form of the model (e.g., linear, nonlinear, or stochastic), characteristic of the decision space (e.g., finite or infinite), or solution process (e.g., prior specification of preferences or interactive). Among MODM methods, we can count multi-objective linear programming (MOLP) and its variants such as multi-objective stochastic integer linear programming, interactive MOLP, and mixed 0-1 MOLP; multi-objective goal programming (MOGoP); multi-objective geometric programming (MOGeP); multi-objective nonlinear fractional programming; multi-objective dynamic programming; and multi-objective genetic programming. The formulations of these programming techniques under fuzziness will not be given here since most of them will be explained in detail in the subsequent chapters of this book with numerical examples. The intelligent fuzzy multi-criteria decision making methods will be explained by Waiel F. Abd El-Wahed in Chapter 2.

When a MODM problem is being formulated, the parameters of objective functions and constraints are normally assigned by experts. In most real situations, the possible values of these parameters are imprecisely or ambiguously known to the experts. Therefore, it would be more appropriate for these parameters to be represented as fuzzy numerical data that can be represented by fuzzy numbers.

4. DIFFUSION OF FUZZY SETS INTO MULTI-CRITERIA DECISION MAKING

The classic MADM methods generally assume that all criteria and their respective weights are expressed in crisp values and, thus, that the rating and the ranking of the alternatives can be carried out without any problem. In a real-world decision situation, the application of the classic MADM method may face serious practical constraints from the criteria perhaps containing imprecision or vagueness inherent in the information. In many cases, performance of the criteria can only be expressed qualitatively or by using linguistic terms, which certainly demands a more appropriate method.

The most preferable situation for a MADM problem is when all ratings of the criteria and their degree of importance are known precisely, which makes it possible to arrange them in a crisp ranking. However, many of the decision making problems in the real world take place in an environment in which the goals, the constraints, and the consequences of possible actions are not known precisely (Bellman and Zadeh, 1970). These situations imply that a real decision problem is very complicated and thus often seems to be little suited to mathematical modeling because there is no crisp definition (Zimmermann and Zysno, 1985). Consequently, the ideal condition for a classic MADM problem may not be satisfied, in particular when the decision situation involves both fuzzy and crisp data. In general, the term “fuzzy” commonly refers to a situation in which the attribute or goal cannot be defined crisply, because of the absence of well-defined boundaries of the set of observation to which the description applies.

A similar situation is when the available information is not enough to judge or when the crisp value is inadequate to model real situations. Unfortunately, the classic MADM methods cannot handle such problems effectively, because they are only suitable for dealing with problems in which all performances of the criteria are assumed to be known and, thus, can be represented by crisp numbers. The application of the fuzzy set theory in the field of MADM is justified when the intended goals or their attainment cannot be defined or judged crisply but only as fuzzy sets (Zimmermann, 1987). The presence of fuzziness or imprecision in a MADM problem will obviously increase the complexity of the decision situation in many ways. Fuzzy or qualitative data are operationally more difficult to manipulate than crisp data, and they certainly increase the computational requirements in particular during the process of ranking when searching for the preferred alternatives (Chen and Hwang, 1992).

Bellman and Zadeh (1970) and Zimmermann (1978) introduced fuzzy sets into the MCDM field. They cleared the way for a new family of methods to deal with problems that had been inaccessible to and unsolvable with standard MCDM techniques. Bellman and Zadeh (1970) introduced the first approach regarding decision making in a fuzzy environment. They suggested that fuzzy goals and fuzzy constraints could be defined symmetrically as fuzzy sets in the space of alternatives, in which the decision was defined as the confluence between the constraints to be met and the goals to be satisfied. A maximizing decision was then defined as a point in the space of alternatives at which the membership function of a fuzzy decision attained its maximum value.

Baas and Kwakernaak's (1977) approach was widely regarded as the most classic work on the fuzzy MADM method and was often used as a benchmark for other similar fuzzy decision models. Their approach consisted of both phases of MADM, the rating of criteria and the ranking of multiple aspect alternatives using fuzzy sets.

Yager (1978) defined the fuzzy set of a decision as the intersection (conjunction) of all fuzzy goals. The best alternative should possess the highest membership values with respect to all criteria, but unfortunately, such a situation rarely occurs in the case of a multiple attribute decision-making problem. To arrive at the best acceptable alternative, he suggested a compromise solution by proposing the combination of max and min operators. For the determination of the relative importance of each attribute, he suggested the use of the Saaty method through pairwise comparison based on the reciprocal matrix.

Kickert (1978) summarized the fuzzy set theory applications in MADM problems. Zimmermann's (1985, 1987) two books include MADM applications. There are a number of very good surveys of fuzzy MCDM (Chen and Hwang, 1992; Fodor and Roubens, 1994; Luhandjula, 1989; Sakawa, 1993).

Dubois and Prade (1980), Zimmermann (1987), Chen and Hwang (1992), and Ribeiro (1996) differentiated the family of fuzzy MADM methods into two main phases. The first phase is generally known as the rating process, dealing with the measurement of performance ratings or the degree of satisfaction with respect to all attributes of each alternative. The aggregate rating, indicating the global performance of each alternative, can be obtained through the accomplishment of suitable aggregation operations of all criteria involved in the decision. The second phase, the ranking of alternatives, is carried out by ordering the existing alternatives according to the resulted aggregated performance ratings obtained from the first phase.

Some titles among recently published papers can show us the latest interest areas of MADM and MODM. Ravi and Reddy (1999) rank both coking and noncoking coals of India using fuzzy multi-attribute decision making. They use Saaty's AHP and Yager's (1978) fuzzy MADM approach to arrive at the coal field having the best quality coal for industrial use. Fan et al. (2002) propose a new approach to solve the MADM problem, where the decision maker gives his/her preference on alternatives in a fuzzy relation. To reflect the decision maker's preference information, an optimization model is constructed to assess the attribute weights and then to select the most desirable alternatives.

Wang and Parkan (2005) investigate a MADM problem with fuzzy preference information on alternatives and propose an eigenvector method to rank them. Three optimization models are introduced to assess the relative importance weights of attributes in a MADM problem, which integrate subjective fuzzy preference relations and objective information in different ways. Omero et al. (2005) deal with the problem of assessing the performance of a set of production units, simultaneously considering different kinds of information, yielded by data envelopment analysis, a qualitative data analysis, and an expert assessment. Hua et al. (2005) develop a fuzzy multiple attribute decision making (FMADM) method with a three-level hierarchical decision making model to evaluate the aggregate risk for green manufacturing projects.

Gu and Zhu (2006) construct a fuzzy symmetry matrix by referring to the covariance definition of random variables as attribute evaluation space based on a fuzzy decision making matrix. They propose a fuzzy AHP method by using the approximate fuzzy eigenvector of such a fuzzy symmetry matrix. This algorithm reflects the dispersed projection of decision information in general. Fan et al. (2004) investigate the multiple attribute decision making (MADM) problems with preference information on alternatives. A new method is proposed to solve the MADM problem, where the decision maker gives his/her preference on alternatives in a fuzzy relation. To reflect the decision maker's subjective preference information, a linear goal programming model is constructed to determine the weight vector of attributes and then to rank the alternatives.

Ling (2006) presents a fuzzy MADM method in which the attribute weights and decision matrix elements (attribute values) are fuzzy variables. Fuzzy arithmetic operations and the expected value operator of fuzzy variables are used to solve the FMADM problem. Xu and Chen (2007) develop an interactive method for multiple attribute group decision making in a fuzzy environment. The method can be used in situations where the information about attribute weights is partly known, the weights of decision makers are expressed in exact numerical values or triangular fuzzy numbers, and the attribute values are triangular fuzzy numbers. Chen and Larbani (2006) obtain the weights of a MADM problem with a fuzzy decision matrix by formulating it as a two-person, zero-sum game with an uncertain payoff matrix. Moreover, the equilibrium solution and the resolution method for the MADM game are developed. These results are validated by a product development example of nano-materials.

Some recently published papers on fuzzy MODM are given as follows: El-Wahed and Abo-Sinna (2001) introduce a solution method based on the theory of fuzzy sets and goal programming for MODM problems. The solution method, called hybrid fuzzy-goal programming (HFGP), combines and extends the attractive features of both fuzzy set theory and goal programming for MODM problems. The HFGP approach is introduced to determine weights to the objectives under the same priorities as using the concept of fuzzy membership functions along with the notion of degree of conflict among objectives. Also, HFGP converts a MODM problem into a lexicographic goal programming problem by fixing the priorities and aspiration levels appropriately. Rasmy et al. (2002) introduce an interactive approach for solving MODM problems based on linguistic preferences and architecture of a fuzzy expert system. They consider the decision maker's preferences in determining the priorities and aspiration levels, in addition to analysis of conflict among the goals. The main concept is to convert the MODM problem into its equivalent goal programming problem by appropriately setting the priority and aspiration level for each objective. The conversion approach is based on the fuzzy linguistic preferences of the decision maker. Borges and Antunes (2002) study the effects of uncertainty on multiple-objective linear programming models by using the concepts of fuzzy set theory. The proposed interactive decision support system is based on the interactive exploration of the weight space. The comparative analysis of indifference regions on the various weight spaces (which vary according to intervals of values of the satisfaction degree of objective functions and constraints) enables the study of the stability and evolution of the basis that corresponds to the calculated efficient solutions with changes of some model parameters. Luhandjula (1984) used a linguistic variable approach to present a procedure for solving the multiple objective linear fractional programming problem (MOLFPP). Dutta et al. (1992) modified the linguistic approach of Luhandjula such as to obtain an efficient solution to MOLFPP. Stancu-Minasian and Pop (2003) points out certain shortcomings in the work of Dutta et al. and gives the correct proof of theorem, which validates the obtaining of the efficient solutions. We notice that the method presented there as a general one does only work efficiently if certain hypotheses (restrictive enough and hardly verified) are satisfied.

Li et al. (2006) improve the fuzzy compromise approach of Guu and Wu (1999) by automatically computing proper membership thresholds instead of choosing them. Indeed, in practice, choosing membership thresholds arbitrarily may result in an infeasible optimization problem. Although a minimum satisfaction degree is adjusted to get a fuzzy efficient solution, it sometimes makes the process of interaction more complicated. To overcome this drawback, a theoretically and practically more efficient two-phase max–min fuzzy compromise approach is proposed. Wu et al. (2006) develop a new approximate algorithm for solving fuzzy multiple objective linear programming (FMOLP) problems involving fuzzy parameters in any form of membership functions in both objective functions and constraints. A detailed description and analysis of the algorithm are supplied. Abo-Sinna and Abou-El-Enien (2006) extend the TOPSIS for solving large scale multiple objective programming problems involving fuzzy parameters. These fuzzy parameters are characterized as fuzzy numbers. For such problems, the α–Pareto optimality is introduced by extending the ordinary Pareto optimality on the basis of the α–level sets of fuzzy numbers. An interactive fuzzy decision-making algorithm for generating an α–Pareto optimal solution through the TOPSIS approach is provided where the decision maker is asked to specify the degree α and the relative importance of objectives.

5. CONCLUSIONS

The main difference between the MADM and MODM approaches is that MODM concentrates on continuous decision space aimed at the realization of the best solution, in which several objective functions are to be achieved simultaneously. The decision processes involve searching for the best solution, given a set a conflicting objectives, and thus, a MODM problem is associated with the problem of design for optimal solutions through mathematical programming. In finding the best feasible solution, various interactions within the design constraints that best satisfy the goals must be considered by way of attaining some acceptable levels of sets of some quantifiable objectives. Conversely, MADM refers to making decisions in the discrete decision spaces and focuses on how to select or to rank different predetermined alternatives. Accordingly, a MADM problem can be associated with a problem of choice or ranking of the existing alternatives (Zimmermann, 1985).

Having to use crisp values is one of the problematic points in the crisp evaluation process. As some criteria are difficult to measure by crisp values, they are usually neglected during the evaluation. Another reason is about mathematical models that are based on crisp values. These methods cannot deal with decision makers' ambiguities, uncertainties, and vagueness that cannot be handled by crisp values. The use of fuzzy set theory allows us to incorporate unquantifiable information, incomplete information, non obtainable information, and partially ignorant facts into the decision model. When decision data are precisely known, they should not be placed into a fuzzy format in the decision analysis. Applications of fuzzy sets within the field of decision making have, for the most part, consisted of extensions or "fuzzifications" of the classic theories of decision making. Decisions to be made in complex contexts, characterized by the presence of multiple evaluation aspects, are normally affected by uncertainty, which is essentially from the insufficient and/or imprecise nature of input data as well as the subjective and evaluative preferences of the decision maker. Fuzzy sets have powerful features to be incorporated into many optimization techniques. Multiple criteria decision making is one of these, and it is certain that more frequently you will see more fuzzy MCDM modeling and applications in the literature over the next few years.

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INTELLIGENT FUZZY MULTI-CRITERIA DECISION MAKING: REVIEW AND ANALYSIS

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Abstract: This chapter highlights the implementation of artificial intelligence techniques to solve different problems of fuzzy multi-criteria decision making. The reasons behind this implementation are clarified. In additions, the role of each technique in handling such problem are studied and analyzed. Then, some of the future research work is marked up as a guide for researchers who are working in this research area.

Key words: Intelligent optimization, fuzzy multi-criteria decision making, research directions

1. INTRODUCTION

1.1 Mathematical Model of Fuzzy Multi-Criteria Decision Making

Multi-criteria decision making (MCDM) represents an interest area of research since most real-life problems have a set of conflict objectives. MCDM has its roots in late-nineteenth-century welfare economics, in the works of Edgeworth and Pareto. A mathematical model of the MCDM can be written as follows:

$$\underset{s}{\text{Min}} \quad Z = [z_1(x), z_2(x), ..., z_K(x)]^T \tag{1}$$

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where

$$S = \{x \in X \big| Ax \le b, x \in R^n, x \ge 0\}$$

where:

Z(x) = *C x* is the *K*-dimensional vector of objective functions and *C* is the vector of cost corresponding to each objective function,

S is the feasible region that is bounded by the given set of constraints,

A is the matrix of technical coefficients of the left-hand side of constraints,

b is the right-hand side of constraints (i.e., the available resources),

x is the *n*-dimensional vector of decision variables.

When the objective functions and constraints are linear, then the model is a linear multi-objective optimization problem (LMOOP). But, if any objective function and/or constraints are nonlinear, then the problem is described as a nonlinear multi-objective optimization problem (NLMOOP). Since problem (1) is deterministic, it can be solved by using different approaches such as follows:

1. Utility function approach,
2. Interactive programming,
3. Goal programming, and
4. Fuzzy programming.

But, in the real world, the input information to model (1) may be vague, for example, the technical coefficient matrix (*A*) and/or the available resource values (*b*) and/or the coefficients of objective functions (*C*). Also, in other situations, the vagueness may exist, such as the aspiration levels of goals ($z_i(x)$) and the preference information during the interactive process. All of these cases lead to a fuzzy multi-criteria model that can be written as follows:

$$\underset{S}{\text{Min}} \;\; Z \cong [z_1(x), z_2(x), \ldots, z_K(x)]^T \tag{2}$$

where

$$S = \{x \in X \big| \tilde{A}\, x \tilde{\le} \tilde{b},\;\; x \in R^n,\;\; x \ge 0\}\,.$$

This fuzzy model is transformed into crisp (deterministic) by implementing an appropriate membership function. So, the model can be classified into two classes. If any of the objective functions, constraints, and membership functions are linear, then the model will be LFMOOP. But, if any of the objective functions and/or constraints and/or membership functions are nonlinear, then the model is described as NLFMOOP.

Different approaches can handle the solution of problem (2). All of these approaches depend on transforming problem (2) from fuzzy model to crisp model via determining an appropriate membership function that is the backbone of fuzzy programming.

Definition 1.1: Fuzzy set

Let X denote a universal set. Then a fuzzy subset $\tilde{A}$ of X is defined by its membership function:

$$\mu \ddot{A} : X \rightarrow [0,1] \tag{3}$$

That assigns to each element $x \in X$ a real number in the interval [0, 1] and $\mu_{\tilde{A}}(x)$ represents the grade of membership function of x in A.

The main strategy for solving model (2) can be handled according to the following scheme:

Step 1. Examine the type of preference information needed.

Step 2. If a priori articulation of preference information is available use, one of the following programming schemes:

2.1 Fuzzy goal programming,
2.2 Fuzzy global criterion, or
2.3 Another appropriate fuzzy programming technique.
Otherwise, go to step (3).

Step 3. If progressive articulation of preference information is available, use the following programming scheme:

3.1 Fuzzy interactive programming,
3.2 Interactive fuzzy goal programming, or
3.3 Another appropriate fuzzy interactive programming technique.

Step 4. End strategy.

Each programming scheme involved different solution methodologies that will be indicated in Section 1.3.

1.2 Historical Background of Fuzzy MCDM

In 1970, Bellman and Zadah highlighted the main pillar of fuzzy decision making that can be summarized as follows:

$$D = G \cap C \tag{4}$$

where G is the fuzzy goal, C is the fuzzy constraints, and D is the fuzzy decision that is characterized by a suitable membership function as follows:

$$\mu_D(x) = \min(\mu_G(x), \mu_C(x)). \tag{5}$$

The maximizing decision is then defined as follows:

$$\max_{x \in X} \mu_D(x) = \max_{x \in X} \min\,(\mu_G(x), \mu_C(x)). \tag{6}$$

For k fuzzy goals and m fuzzy constraints, the fuzzy decision is defined as follows:

$$D = G_1 \cap G_2 \cap \ldots \cap G_k \cap C_1 \cap C_2 \cap \ldots \cap C_m \tag{7}$$

and the corresponding maximizing decision is written as follows:

$$\max_{x \in X} \mu_D(x) = \max_{x \in X} \min\,(\mu_{G_1}(x),\ldots, \mu_{G_k}, \mu_{C_1}(x)\ldots, \mu_{Cm}(x)). \tag{8}$$

For more details about this point, see Sakawa (1993). Since this date, many research works have been developed. In this section, the light will be focused on a sample of research works on FMCDM from the last 25 years to extract the main shortcomings that argue for us to direct attention toward the intelligent techniques as an alternative methodology for overcoming these drawbacks.

In FMCDM problems, the membership function depends on where the fuzziness existed. If the fuzziness in the objective functions coefficients, the membership function may be represented by

$$\mu_k(Z^k(x)) = \begin{cases} 1 & \text{if} \quad Z^k(x) \le L_k, \\ \dfrac{U_k - Z^k(x)}{U_k - L_k} & \text{if} \quad L_k < Z^k(x) < U_k \\ 0 & \text{if} \quad Z^k(x) \ \ge \ U_k \end{cases} \tag{9}$$

where U_k is the worst upper bound and L_k is the best lower bound of the objective function k, respectively. They are calculated as follows:

$$\begin{aligned} U_k &= (Z^k)^{\max} = \max_{x \in X} Z^k(x) \\ L_k &= (Z^k)^{\min} = \min_{x \in X} Z^k(x), \ k = 1, 2, \ldots, K \end{aligned} \tag{10}$$

If the fuzziness is existed in the right-hand side of the constraints, the constraints are transformed into equalities and then the following membership function is applied (Lai and Hwang, 1996):

$$\mu_k(Z^k(x)) = \begin{cases} [(Ax)_i - (b_i - d_i)]/d_i & \textit{if} \quad (b_i - d_i) \le (Ax)_i < b_i, \\ [(b_i - d_i) - (Ax)_i]/d_i & \textit{if} \quad b_i \le (Ax)_i \le (b_i - d_i) \\ 0 & \textit{if} \quad (b_i - d_i) \le (Ax)_i or (Ax)_i < (b_i - d_i) \end{cases} \tag{11}$$

where the membership function is assumed to be symmetrically triangular functions. The problem solver may assume any other membership function based on his/her experience. Besides, some mathematical and statistical methods develop a specific membership function. On the other side, the intelligent techniques provide the problem solver with a powerful techniques to create or estimate these functions as will be indicated later. If we assumed that the FMCDM problem has fuzzy objective functions, then the deterministic model of the FMCDM is written as follows:

$$\begin{aligned} &\max \ \beta \\ &\text{subject to} \\ &\beta \le \mu_k(Z^k(x)), \ k = 1,2,\ldots,K \\ &\sum_{j=1}^{n} a_{ij} x_j = b_i, \ i = 1,2,\ldots,m \\ &x_j \ge 0, \ i = 1,2,\ldots,m; \ j = 1,2,\ldots,n, \ k = 1,2,\ldots,K \\ &0 \le \beta \le 1 \end{aligned} \tag{12}$$

where β is an auxiliary variable and can be worked at a satisfaction level. Model (7) can be solved as a single objective linear/nonlinear programming problem.

After the Bellman and Zadah paper, several research studies were adopted, such as Hannan (1983) and Zimmerman (1987) who handled fuzzy linear programming with multiple objectives by assuming a special form of the membership function. Hannan assumed discrete membership function, and Zimmerman used a continuous membership function. Boender (1989), Sakawa (1993), and Baptistella and Ollero (1980) implemented the fuzzy set theory in interactive multi-criteria decision making. For more historical information, see Sakawa (1993) and Lai and Hwang (1996). Also, see Biswal (1992), Bhattacharya et al. (1992), Bit (1992), Boender et al. (1989), Buckley (1987), Lothar and Markstrom (1990) for more solution methodologies.

Many real-life problems have been formulated as FMCDM and have been solved by using an appropriate technique. Some of these applications involved production, manufacturing, location−allocation problems, environmental management, business, marketing, agriculture economics, machine control, engineering applications and regression modeling. A good classification with details can be found in Lai and Hwang (1996). A new literature review (Zopounidis and Doumpos, 2002) assures the same field of applications.

1.3 Shortcomings of the FMCDM Solution Approaches

The problems that meet either the solution space construction or the model development can be classified into three categories as follows: 1) ill-structured, 2) semi-well structured or, 3) well structured.

Each category has been characterized by specific criteria to indicate its class. Some of these indicator criteria of ill-structured problems are as follows:

1. There is no available solution technique to solve the model.
2. There is no standard mathematical model to represent the problem.
3. There is no ability to involve the qualitative factors in the model.
4. There is no available solution space to pick up the optimal solution.
5. There is a difficulty in measuring the quality of the result solution(s).
6. There is kind of vagueness of the available information that leads to complexity in considering it into the model account.

If some of these criteria exist, then the problem will belong to the second category, which is called *semi-ill-structured* problems. But, if all of these criteria and others do not exist, then the problem will belong to the third category, which is called *well-structured* problems. It is clear that there is no problem regarding the third category. Fortunately, the first and second categories represent a rich area for investigation, especially in the era of information technology where all the sciences are interchanged in a complex manner to a degree that one can find difficulty in separating between sciences. In other words, biological sciences, sociology, insects' science, and so on attracted researchers to simulate them by using computer technology that consequently reflects its positive progress on the optimization research work.

Let us now apply these criteria of ill-structured problems on FMCDM problems. For FMCDM model structure, the following problems are represented as an optical stone to more progress in this area. Some of these problems are as follows:

1. Incorporating fuzzy preferences in the model still needs new methodologies to take the model into account without increasing the model complexity.
2. Right now, the FMCDM models are transformed into crisp models to solve it by using the available traditional techniques. This transformation reduces both the efficiency and the effectiveness of the fuzzy solution methodologies. So, we need to look for a new representation methodology to increase or at least keep the efficiency of the fuzzy methodology.
3. As mentioned above, the membership function is the cornerstone of fuzzy programming, and right now, the problem solvers assumed it according the experience. As a result, the solution will be different according to the selected membership function. This will lead to another problem, which is which solution is better or qualifies more for the problem under study. In this case, there is an invitation to implement the progress in information technology to discover an appropriate membership function.
4. Large-scale FMCDM models still need more research especially when incorporating large preference information.

Regarding the solution methodologies, there are some difficulties in enhancing them. Some of them are:

1. Some of the existing ranking approaches that have been used to solve the FMCDM problem are not perfect.

2. Fuzzy integer programming with multi-criteria can be considered a combinatorial optimization problem, and as a result, it needs an exponential time algorithm to go with it.
3. In 0-1 FMCDM problems (whatever small scale or large scale), the testing process of the Pareto-optimal solution is considered the NP-hard problem.
4. In FMCDM problems, a class of problems exist that are known as the global convex problems, where the good solutions in the objective space are similar to those in the decision space. So, we need a new methodology to perform well with them.
5. In fuzzy and nonfuzzy MCDM problems, there is a difficulty in constructing an initial solution that should be close to the Pareto-optimal solution to reduce the solution time. So, we need powerful methodology−based information technology to deal with this problem.

Because of these shortcomings and others, FMCDM attracts the attentions of researchers to enhance the field of FMCDM by developing more powerful links (bridges) between it and other sciences. In this chapter, we will highlight the link between artificial intelligence and FMCDM to overcome all or some of the mentioned problems. This link leads to a new and interesting area of research called "intelligent optimization." The general strategy for the integration between artificial intelligence (AI) techniques and FMCDM problems may be done according the following flowchart seen in Figure 2. In the next subsection, some of the intelligent techniques will be introduced briefly.

1.4 Some Intelligent Techniques

AI is the branch of computer technology that simulates the human behavior via intelligent machines to perform well and better than humans. Computer science researchers are wondering how to extract their ideas from the biological systems of human beings such as thinking strategies, the nervous system, and genetics. AI also extends to the kingdom of insects such as the ant colony. The tree that summarizes the different commercial forms of AI techniques is shown in Figure 1. Each AI technique can perform well in specific situations more so than in others. For example, expert systems (ESs) can handle the qualitative factors or preferences that can not be included in the mathematical model. Artificial neural networks (ANNs) are successfully applied in prediction, classification, pattern and voice recognition, and so on. Simulated annealing

(SA), genetic algorithms (GA), and particle swarm optimization (PSO) are used as stochastic search methods to deal with multi-criteria combinatorial optimization problems.

The implementation of AI techniques to handle different problems in FMCDM depends on the following conditions:

1. The nature of the problem that FMCDM suffers from,
2. The availability of the solution techniques and its performance,
3. The environmental factors that affect the problem under study.

AI techniques can be classified according to their functions as follows:

1. Symbolic processing, where the knowledge is treated symbolically not numerically. In other words, the process is not algorithmical. These techniques are ES, fuzzy expert system (FES), and decision support system (DSS).
2. Search methods that are implemented to search and scan the large solution space of combinatorial optimization problem. These techniques are able to pick an acceptable or preferred solution in less time compared with the traditional solution procedures. Examples of these search methods are GA, SA, ant colony optimization (ACO), PSO, DNA computing, and any hybrid of them.
3. Learning process that is responsible for doing forecasting, classifications, and function estimating based on enough historical data about the problem under study. These techniques are ANN and neuro-fuzzy systems.

Now, we shall classify the intelligent FMCDM problems based upon the implemented technique.

1.4.1 Expert System and FMCDM

ES is an intelligent computer program that consists of three modules: 1) inference engine module, 2) knowledge-base module, and 3) user-interface module. This system can produce one of the following functions: 1) conclusion, 2) recommendation, and 3) advice. The main feature of the ES is its ability to treat the problems symbolically not algorithmically. So, it can perform a good job regarding both the decision maker's preferences and the qualitative factors that cannot be included in the mathematical model because of its increase in the degree of model complexity.

Figure 1. The tree diagram of artificial intelligence techniques

Generally speaking, ES has been applied to solve different applications that can be modeled in MCDM. For example, Lothar and Markstrom (1990) developed an expert system for a regional planning system to optimize the industrial structure of an area. In this system, AI paradigms and numeric multi-criteria optimization techniques are combined to arrive at a hybrid approach to discrete alternative selection. These techniques include 1) qualitative analysis, 2) various statistical checks and recommendations, 3) robustness and sensitivity analysis, and 4) help for defining acceptable regions for analysis. Jones et al. (1998) developed an intelligent system called "GPSYS" to deal with linear and integer goal programming. The intelligent goal programming system is one that is designed to allow a nonspecialist access to, and clear understanding of a goal programming solution and analysis techniques. GPSYS has an analysis tool such as Pareto

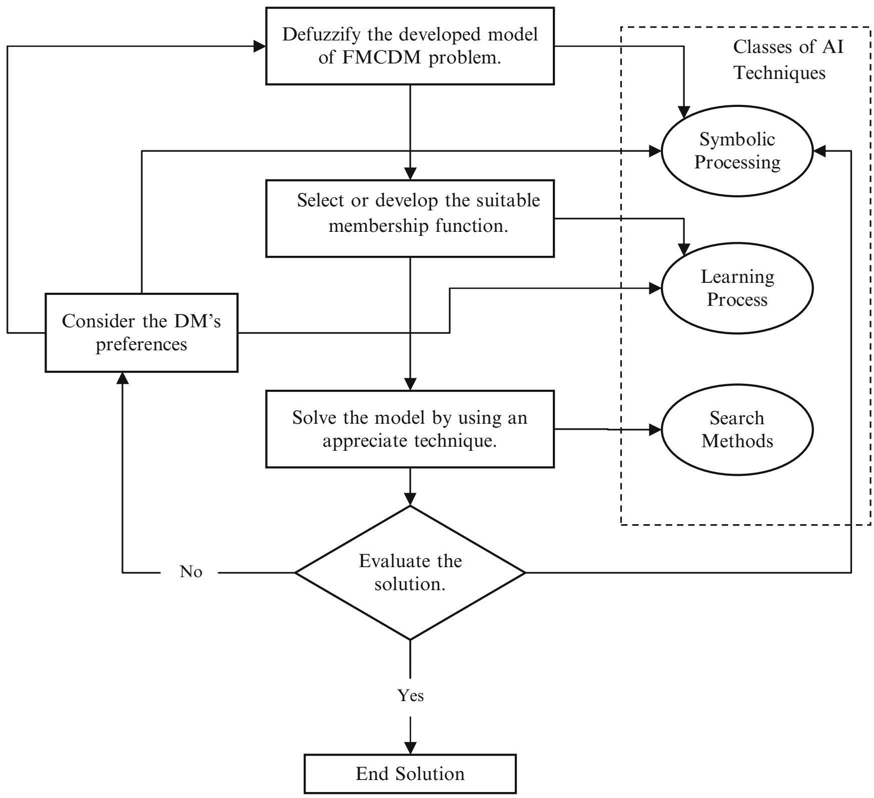

Figure 2. The integration between AI techniques and FMCDM phases

detection and restoration, normalization, automated lexicographic redundancy checking, and an interactive facility. Abd El-Wahed (1993) developed a decision support system with a goal programming based ES to solve engineering problems. In this research, the statistical analysis and the decision maker's preferences are combined in an ES to assign the differential weights of the sub-goals in goal programming problems. Also, Rasmy et al. (2001) presented a fuzzy ES to include the qualitative factors that could not be involved in the mathematical model of the multi-criteria assignment problem in the field of bank processing. The approach depends on evaluating the model solution by using the developed fuzzy ES. If the solution is coincided with the evaluation criteria, the approach is terminated. Otherwise, some modification on the preferences is done in the feedback to resolve the model again and so on until getting a solution coincides with the evaluation criteria. Little research work regarding FMCDM has been done. For example, Rasmy et al. (2002) presented an interactive approach for solving the MCDM problem with fuzzy preferences in both aspiration level determination and priority structure by using the framework of the fuzzy expert system. The main idea of this approach is to convert the MCDM problem into its equivalent goal programming model by setting the aspiration levels and priority of each objective function based on fuzzy linguistic variables. This conversion makes the implementation of ES easy and effective.

Liu and Chen (1995) present an integrated machine troubleshooting expert system (IMTES) that enhances the efficiency of the diagnostic process. The role of fuzzy multi-attribute decision-making in ES is determined to be the most efficient diagnostic process, and it creates a "meta knowledge base" to control the diagnosis process.

The results of an update search in some available database sites regarding the combination of both ES and FMCDM can be summarized as follows:

1. The mutual integration between ES and MCDM/FMCDM is a rich area for more research,
2. The implementation of ES for dealing with the problems of FMCDM still needs more research,
3. The combination of ES and other AI techniques needs more research to gain the advantages of both of them in solving the problems of FMCDM problems.

The researchers are invited to investigate the following points where they are not covered right now:

1. Applying ES to guide the determination process of the aspiration levels of fuzzy goal programming.
2. Applying ES to handle the DM's preferences in solving interactive FMCDM to reduce the solution time and the solution efforts.
3. Implementing the ES in ranking approaches that have been used to solve FMCDM problems to include the environmental qualitative factors.
4. Handling ES in solving large-scale FMCDM problems.
5. Combining ES with both parametric analysis and sensitivity analysis to pick a more practical solution.

1.4.2 ANN and FMCDM Problems

ANN is a simulation of a human nervous system. The ANN simulator depends on the Third Law of Newton: "For any action there is an equal reaction with negative direction." A new branch of computer science is opened for research called "neural computing." Neural computing has been viewed as a promising tool to solve problems that involve large date/preferences or what is called in optimization large-scale optimization problems. Also, the transformation of FMCDM into crisp model needs an appropriate membership function. In other situations, ANN is implemented to solve the FMCDM problems without the need to defuzzify the mathematical model of FMCDM problems. ANN offers an excellent methodology for estimating continuous or discrete membership functions/ values. To do that, an enormous amount of historical data is needed to train and test the ANN as well as to get the right parameters and topology of it to solve such a problem. On the other side, the complex combinatorial FMCDM problems (NP hard problems) may be not represented in a standard mathematical form. As a result, ANN can be used to simulate the problem for the purpose of getting an approximate solution based on a simulator. The main problem facing those who are working in this area is the development of the energy (activation) function, which is the central process unit of any ANN. This function should have the inherited characteristics of both the objective function and the constraints to train and test the network. There are many standard forms of it such as the sigmoid function and the hyperbolic function. The problem solver must elect a suitable one from them such that can be fitted with the nature of the problem under study. For the FMCDM with fuzzy objective functions [model (7)], the energy function can be established by using the Lagrange multiplier method as follows:

$$E(x,\beta,\lambda,\eta) = \beta + \lambda^t(-\mu_k(Z^k(x)) + \beta + \chi) + \eta^t(\sum_{j=1}^{n} a_{ij}x_j - b_i) \qquad (11)$$

where λ and η are the Lagrange multipliers. χ is the vector of slack variables. By taking the partial derivative of an equation with respect to x, λ, and η, we obtain the following differential equations:

$$\begin{aligned} \partial E / \partial x &= \rho \nabla_x E(x, \beta, \lambda, \eta) \\ \partial E / \partial \lambda &= \rho \nabla_\lambda E(x, \beta, \lambda, \eta) \\ \partial E / \partial \eta &= \rho \nabla_\eta E(x, \beta, \lambda, \eta) \end{aligned} \tag{12}$$

where ρ is called a learning parameter. By setting the penalty parameters λ and η, the adaptive learning parameters ρ, and initial solution $x_j(0)$, then we can solve the system (9) to obtain β.

Previous research works use ANN to solve some optimization problems as well as FMCDM specifically. These works can be classified according to the type of treating method of the FMCDM model as follows:

1.4.2.1 Treating the Fuzzy Preferences in MCDM Problems

For example, Wang (1993) presented a feed-forward ANN approach with a dynamic training procedure to solve multi-criteria cutting parameter optimization in the presence of fuzzy preferences. In this approach, the decision maker's preferences are modeled by using fuzzy preference information based on ANN. Wang and Archer (1994) modeled the uncertainty of multi-criteria, multi-persons decision making by using fuzzy characteristics. They implemented the back propagation learning algorithm under monotonic function constraints. Stam et al. (1996) presented two approaches of ANNs to process the preference ratings, which resulted from analytical, hierarchy process, pair-wise comparison matrices. The first approach, implements ANN to determine the eigenvectors of the pair-wise comparison matrices. This approach is not capable of generalizing the preference information. So, it is not appropriate for approximating the preference ratings if the decision maker's judgments are imprecise. The second approach uses the feed-forward ANN to approximate accurately the preference ratings. The results show that this approach is working well with respect to imprecise pair-wise judgments. Chen and Lin (2003) developed the decision neural network (DNN) to use in capturing and representing the decision maker's preferences. Then, with DNN, an optimization problem is solved to look for the most desirable solution.

1.4.2.2 Handling Fuzziness in FMCDM Models

It is clear that ANN is capable of solving the constrained optimization problems, especially the applications that require on-line optimization. Gen et al. (1998) discussed a two-phase approach to solve MCDM problems with fuzziness in both objectives and constraints. The main proposed steps to solve the FMCDM model (2) can be summarized as follows:

1. Construct the membership function based on positive ideal and negative ideal (worst values) solutions.
2. Apply the concept of α-level cut, where $\alpha \in [0,1]$ to transform the model into a crisp model.
3. Develop the crisp linear programming model based on steps (1) and (2).
4. According to the augmented Lagrange multiplier method, we can create the Lagrangian function to transform the result model in step (3) into an unconstrained optimization problem. The Lagrangian function is implemented as an energy (activation) function to activate the developed ANN.
5. If the DM accepts the solution, stop. Otherwise, change α and go to the step (1).

The results show that the result solution is close to the best compromise solution that has been calculated from the two-phase approach. The method has an advantage; if the decision maker is not satisfied with the obtained solutions, he/she can get the best solutions by changing the α-level cut.

1.4.2.3 Determining the Membership Functions

Ostermark (1999) proposed a fuzzy ANN to generate the membership functions to new data. The learning process is reflected in the shape of the membership functions, which allows the dynamic adjustment of the functions during the training process. The adopted fuzzy ANN is applied successfully to multi-group classification-based multi-criteria analysis in the economical field.

1.4.2.4 Searching the Solution Space of Ill-Structured FMCDM Problems

Gholamian et al. (2005) studied the application of hybrid intelligent system based on both fuzzy rule and ANN to:

- Guide the decision maker toward the noninferior solutions.
- Support the decision maker in the selection phase after finishing the search process to analyze different noninferior points and to select the best ones based on the desired goal levels.

The idea behind developing this system is the ill-structured real-world problem in marketing problems where the objective can not be expressed in a mathematical form but in the form of a set of historical data. This means that ANN can do well with respect to any other approach. From the above analysis, we can deduce that many research points are still uncovered. It means that the integration area between ANN and FMCDM is very rich for more research. These points are summarized as follows:

1. Applying the ANN to solve FMCDM problems in its fuzzy environment without transforming it into a crisp model to obtain more accurate, efficient, and realistic solution(s).
2. Developing more approaches to enhance the process of generating real membership functions.
3. Studying the effect of using different membership functions on the solution quality and performance.
4. Implementing the ANN to solve more large-scale FMCDM problems that represented the real-life case.
5. Combining both ES and ANN to develop more powerful approaches to consider the preference information (whatever quantitative/qualitative) in FMCDM problems.
6. Applying the ANN to do both parametric and sensitivity analysis of the real-life problems that can be represented by the FMCDM model.

1.4.3 Tabu Search

A tabu search (TS) was initiated by Glover as an iterative intelligent search technique capable of overcoming the local optimality when solving the CO problems. The search process is based on a neighborhood mechanism. The neighborhood of a solution is defined as a set of all formations that can be obtained by a move that is a process for transforming the search from the current solution to its neighboring solution. If the move is not listed on the TS, the move is called an “admissible move.” If the produced solution at any move is better than all enumerated solutions in prior iterations, then this solution is saved as the best one. The candidate solutions, at each iteration, are checked by using the following tabu conditions:

1. Frequency memory that is responsible for keeping the knowledge of how the same solutions have been determined in the past.
2. Recency memory that prevents cycles of length less than or equal to a predetermined number of iterations.

TS has an important property that enables it to avoid removing the powerful solutions from consideration. This property depends on an element called an aspiration mechanism. This element means that if the TS list captured a solution with a value strictly better than the best obtained so far, the TS can stop.

TS is applied to solve some FMCDM problems. For example, Bagis (2003) proposed a new approach based on TS to determine the membership functions of a fuzzy logic controller. The simulation results indicated that the given approach is performed well, and as a result it is effective in determining such a membership function. Li et al. (2004) presented a TS method as a stochastic global optimization method for solving very large combinatorial optimization tasks and for extending a continuous-valued function for the fuzzy optimization problems. They approved the performance of the proposed method by applying it to an elementary fuzzy optimization problem such as the method for fuzzy linear programming; fuzzy regression and the training of fuzzy neural networks are also presented. Choobineh et al. (2006) proposed an algorithm to deal with a sequencing of n-jobs on a single machine with sequence-dependent setup times and m-objective functions. The algorithm generates a set of solutions that reflects the objectives' weights and close to the best observed values of the objectives. In addition, the authors formulated a mixed integer linear program to obtain the optimal solution of a triple-objective functions problem. Most of the published research works have not focused on FMCDM problems.

1.4.4 Simulated Annealing (SA)

The SA algorithm is a search technique designed to look for a global minimum among many local minima. The algorithm simulates the thermodynamic process of annealing metals by slow cooling where at high temperatures, molecules in metal move rapidly with respect to each other. If the metal is slow cooled sufficiently, then thermal mobility is lost. The resulting arrangement of atoms tends to form a pure crystal that is completely ordered. This ordered state occurs when the system has achieved minimum energy by an annealing process that must be cooled sufficiently slowly to reach thermal equilibrium.

The SA search method is a powerful tool to provide excellent solutions of single objective optimization problems to reduce the computational cost. Later, this approach was adapted for the multi-objective framework by Serafini (1985), Czyżak et al. (1994) and Ulungu et al. (1995). But they examined only the notion of the probability in the multi-objective framework. Serafini (1985) used simulated annealing on the multi-objective framework. Czyżak and Jaszkiewicz (1998) and Ulungu et al. (1998) designed a complete MOSA algorithm and tested it with a multi-objective combinatorial optimization problem. Ulungu et al. (1999) presented an interactive version of MOSA to solve an industrial application problem. Suppapitnarm et al. (2000) proposed a different simulated annealing approach to handle multi-objective problems. Czyżak et al. (1994) hybridized both SA and GA to provide efficient solutions of multi-objective optimization problems. Loukil et al. (2006) proposed a multi-objective SA algorithm to tackle a production scheduling problem in a flexible job-shop with particular constraints such as batch production; production of several sub-products followed by assembly of the final product, and possible overlaps for the processing periods of two successive operations of the same job. For more details in this area of research, see both Suman (2002) and (2003).

In the literature, there are some research works regarding MCDM problems, and the available fuzzy research works are under the general title "fuzzy optimization" not specific FMCDM problems. So, this area of research is ripe for more investigations.

1.4.5 Genetic Algorithms and FMCDM

The GA is a search algorithm that mimics the processes of natural evolution. The problem addressed by GA is searching the solution space is to identify the best problems that are combinatorial or large scale or ill-structured in general. GA encodes the variables of problems in either binary or real-valued vectors. Each code is called a chromosome. In binary coding there are two decoding functions to convert from real to binary and vice versa. In addition, mutation, crossover, and selection are the three important operators used for generating a new solution within the solution space. For example, the mutation operator introduces new genetic material into the population. Crossover recombines individuals to create new individuals. The selection process elects the next generation by using 1) tournament selection, 2) proportional selection, 3) ranking selection, 4) steady-state selection, and 5) manual selection. An evaluation function called the "fitness function" is generated to test the result solution. In the case of constrained optimization problems, Lagrange multipliers are used to transform the problem into an unconstrained optimization problem to be used as a fitness function. The general flowchart of a GA for solving an optimization problem is shown in Figure 3.

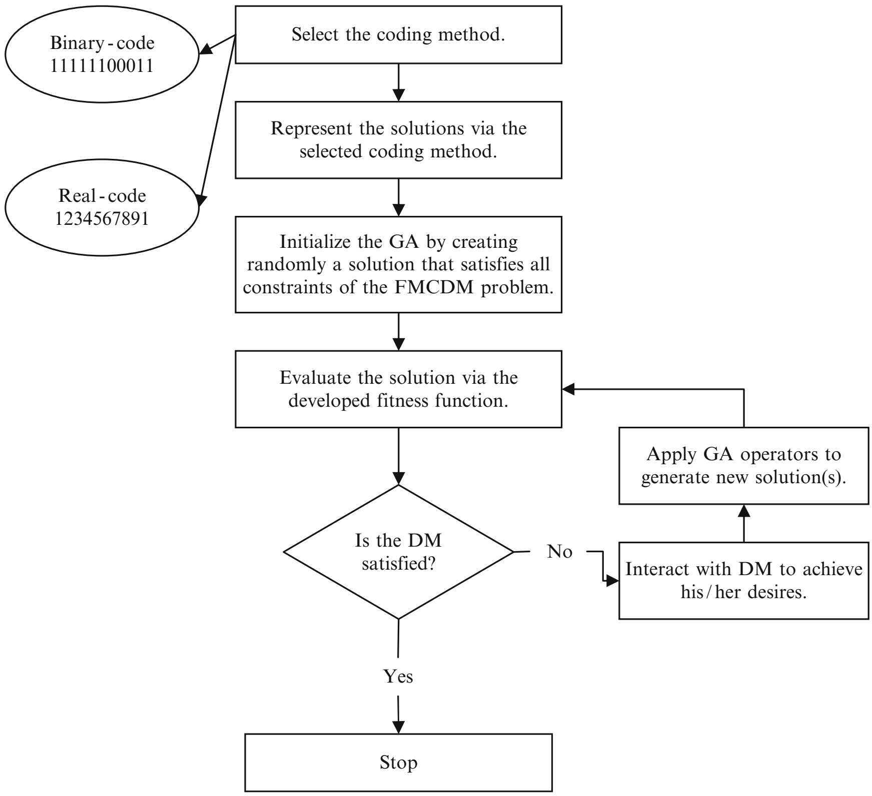

Figure 3. General schema of GA to solve FMCDM problems

GAs seem desirable for solving MOOPs because they deal simultaneously with a set of solutions (the so-called population) that allows the problem solver to find several members of the Pareto optimal set in a single run of the algorithm, instead of having to perform a series of separate runs, such as with the traditional mathematical programming techniques. Additionally, GAs are less susceptible to the shape or continuity of the Pareto front, whereas these two issues are a real concern for mathematical programming techniques. The integration between GA and MOOPs can be classified in the following two categories:

⬣Non-Pareto Techniques

Under this category, we will consider approaches that do not incorporate directly the concept of Pareto optimality. Although these approaches are efficient, most of them enable us to produce certain portions of the Pareto front. However, their simplicity has made them popular among a certain sector of researchers. These approaches are as follows:

1. Aggregating approaches,
2. Lexicographic ordering,
3. The ε-constraint method, and
4. Target-vector approaches.

●Pareto-Based Techniques

In this category, the main idea is finding the set of strings in the population that are Pareto nondominated by the rest of the population. These strings are assigned the highest rank and are eliminated from additional considerations. Another set of Pareto nondominated strings are determined from the remaining population and are assigned the next highest rank. Some of the approaches that implement this idea are:

1. Pure Pareto ranking,
2. Multi-objective genetic algorithm (MOGA),
3. Nondominated sorting genetic algorithm (NSGA), and
4. Nondominated pareto genetic algorithm (NPGA).

In the context of this chapter, some works have been found and can be classified into the following categories:

1.4.5.1 Interactive FMCDM-Based GA

Sakawa and others presented a series of papers in this category. The ideas of these works can be summarized in the following:

- Kato et al. (1997) introduce an interactive satisfying method using GA for getting the satisfying solution for a decision maker from an extended Pareto optimal solution set. In this method, for a certain value of α-level cut and reference membership function, the solution of large-scale multi-objective 0-1 programming is obtained by adopting a GA with decomposition procedures.
- Sakawa and Yauchi (1999) highlight the multi-objective, nonconvex, nonlinear programming problems with fuzzy goals and solve it by applying an interactive fuzzy satisfying method. In this method, the Pareto optimal solution is obtained by solving the augmented mini-max problem for which the floating point GA called GENOCOP III is applicable.

- Sakawa and Yauchi (2000) proposed an interactive decision-making method for solving multi-objective, nonconvex programming problems with fuzzy numbers through co-evolutionary GAs. In this paper, the authors were trying to overcome the drawbacks of GENCOP III by introducing a method to generate an initial feasible point and a bisection method. This modification leads to a new GENCOP called revised GENCOP III.
- Sakawa and Kubota (2000) solved an application in job shop scheduling with fuzzy processing time and fuzzy due date by using GA.
- Sakawa and Kato (2002) deal with the general multi-objective 0-1 programming problems that involve positive and negative coefficients. The extended GA with double strings is implemented with a new decoding algorithm for individuals. The double strings map each individual to a feasible solution based on backtracking and individual modification. For more details about the GA and FMCDM, see Sakawa (2002).
- Basu (2004) applied an interactive fuzzy satisfying method based on an evolutionary programming technique for short-term multi-objective hydrothermal scheduling. The multi-objective problem is formulated by assuming that the decision maker has fuzzy goals for each of the objective functions and that the evolutionary programming technique-based fuzzy satisfying method is applied for generating a corresponding optimal noninferior solution for the decision maker's goals.
- Wahed et al. (2005) presented a contribution in this area by suggesting an interactive approach to determine the preferred compromise solution for the MCDM problems in the presence of fuzzy preferences. Here, the decision maker evaluates the solution by using a defined set of linguistic variables, and consequently, the achievement membership function can be constructed for each objective function. The used non-negative differential weights are determined based on the entropy degree of each objective function to support transforming the MCDM into a single objective function.

1.4.5.2 Goal Programming-Based GAs

Goal programming (GP) is an important technique that is capable of solving a problem with multiple goals. The concept of goal programming (GP) is extended to solve multi-objective decision-making problems because of its ability to transform it into a single-objective programming problem with or without priority through putting the objective functions as goal constraints with predetermined aspiration levels. Also, FGP is extended to solve the complex problems in MCDM/FMCDM problems, especially with implementing GAs. In this case, some research works have been enumerated as follows:

- Zheng et al. (1996) discussed the initialization process, fitness function structure, and the GA operators in the proposed GA for solving nonlinear goal programming (NLGP).
- Gen et al. (1997) developed a GA to solve fuzzy NLGP. They assumed that the implemented membership functions are strictly monotone decreasing (or increasing) and continuous functions with the set of objective functions and certain maximum tolerance limits to the given resources.
- Hu et al. (2007) suggested a method for generating the solution that is consistent with the decision maker's desires where the goal with high priority may have the first level of goal achievement. The method uses a co-evolutionary genetic algorithm to solve the nonlinear, nonconvex problem that results from the original problem. GENCOPIII package is used to handle this problem.

1.4.5.3 Fuzzy Programming-Based GAs

- Li et al. (1997) presented an improved GA for solving a multi-objective solid transportation problem with consideration of the coefficients of the objective function as fuzzy numbers. The selection and evaluation process in GA are done by incorporating ranking of fuzzy numbers with integral value.
- Kim (1998) designed a two-phase genetic algorithm to improve the system performance in nonlinear and complex problems. The first phase is responsible for generating a fuzzy rule base that covers as many of the training examples as possible. The second phase constructed fine-tuned membership functions that minimize the system error.
- Liu and Iwamura (2001) provide a fuzzy simulation-based GA to handle both fuzzy objectives and goal constraints as well as other ideas.
- Jimenez et al. (2003) proposed an evolutionary algorithm to solve fuzzy nonlinear programming as a first step to solving the general nonlinear programming problem.
- Sasaki and Gen (2003) proposed a GA for solving fuzzy multiple objective design problems by implementing a new chromosomes representation that makes the GA more effective.
- Wang et al. (2005) implemented the multi-objective GA to extract interpretable fuzzy rule-based knowledge from data where the genes

are arranged into control genes and parameter genes. This division enables the fuzzy sets and rules to be optimally reduced.

At the end of this section, we can decide that the implementation of GAs in solving the FMCDM problems are occupied a wide interest of the research move so than any other AI searches technique. For more knowledge, see the following website: http://www.jeo.org/emo/EMOOjournals.html. However, there are still some problems in FMCDM problems that have not been studied yet such as:

1. Large-scale FMCDM problems with fuzzy numbers in the objective functions and constraints.
2. Combining both ES and GA to handle the fuzzy preferences in MCDM problems to get a more powerful solution method.
3. Implementing the GA to study both sensitivity and parametric analysis of linear and nonlinear FMCDM.

1.4.6 Ant Colony Optimization

Ant colony optimization (ACO) is a meta-heuristic approach that emulates the foraging behavior of real ants to find the shortest paths between food sources and their nest. This approach is proposed by Dorigo (1992). During the ant's walk from food sources and vice versa, ants deposit a chemical substance called "*Pheromone*" on the ground to guide the rest of ants to the shortest and safest path they should follow. The artificial ants that simulate the real ants perform random walks on a completely connected graph $G = (S, L)$, whose vertices are the solution components S and the connections L. This graph is based on probabilistic model called the "Pheromone model." When a constrained combinatorial optimization problem is considered, the constraints are built into the ants to get the feasible solution(s) only. ACO methods have been successfully applied to diverse combinatorial optimization problems, including traveling salesman, quadratic assignment, vehicle routing, telecommunication networks, graph coloring, constraint satisfaction, Hamiltonian graphs, and scheduling (Cordon et al., 2002). The following chart indicated the mechanism of ACO in solving combinatorial optimization (CO).

The ACO approach is performing well in combinatorial network optimization problems where the solution space is difficult to enumerate especially in large-scale problems. It has been applied to solve the multi-objective combinatorial optimization problems. For example, Chan and Swarnkar (2006) present a fuzzy goal programming approach to model the

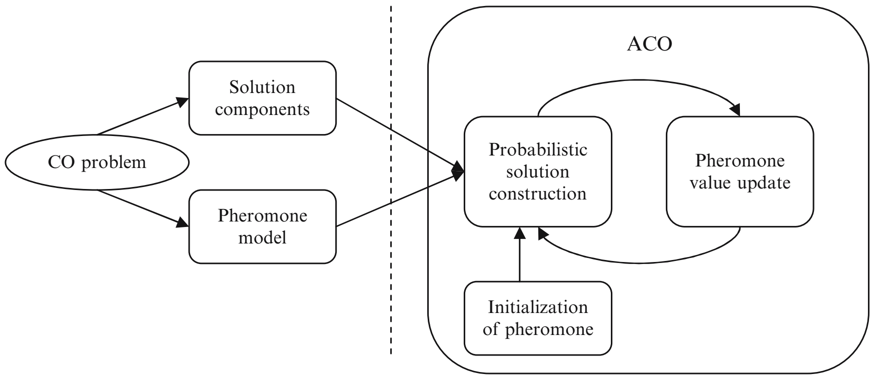

Figure 4. Mechanism of ACO in solving combinatorial optimization (Blum, 2005)

machine tool selection and operation allocation problem of flexible manufacturing systems. The proposed model is optimized by an ant colony algorithm to the computational complexities involved in solving the problem. Doerner et al. (2006) applied Pareto ant colony optimization (P-ACO) that performs particularly well for integer linear programming. The given procedure identifies several efficient portfolio solutions within a few seconds and correspondingly initializes the pheromone trails before running P-ACO. This extension offers a larger exploration of the search space at the beginning of the search with low cost. Marc Gravel et al. (2002) applied the ACO for getting the solution of an industrial scheduling problem in an aluminum casting center. They present an efficient representation scheme of a continuous horizontal casting process that takes into account several objectives that are important to the scheduler.

A little research work has been done in using ACO and MCDM/ FMCDM problems. Most of the research work is done in multi-objective combinatorial optimization problems (MOCOPs) since the meta-heuristics perform much better than the other approaches. So, this area needs more and more research especially in combinatorial FMCDM problems.

1.4.7 Particle Swarm Optimization (PSO)

The basic principles of PSO are represented by a set of moving particles that is initially thrown inside the search space. Each particle is characterized by the following features:

1. A position and a velocity,
2. It knows its position and the objective function value for this position,

3. It knows its neighbors, the best previous position, and the objective function value,
4. It remembers its best previous position,
5. It is considered that the neighborhood of a particle includes this particle itself.

At each time step, the behavior of a given particle is a compromise between three possible choices:

1. Following its own way,
2. Going toward its best previous position,
3. Going toward the best neighbor's best previous position.

The basic equations of PSO can be formalized as follows:

$$\begin{cases} v_{t+1} = c_1 v_t + c_2\left(p_{i,t} - x_t\right) + c_3\left(p_{g,t} - x_t\right) \\ x_{t+1} = x_t + v_{t+1} \end{cases} \tag{13}$$

with

v_t : = velocity at time step t,

x_t : = position at time step t,

P_{it} : = best previous position at time step t,

P_{gt} : = best neighbours previous best, at time step t, (or best neighbor),

c_1, c_2, c_3 := social/cognitive confidence coefficients.

PSO has been used in solving some real-life applications that involved multi-objectives. For example, Parsopoulos and Vrahatis (2002) presented the first study on MCDM by using PSO algorithm. The authors highlighted some important issues such as:

1. The ability of PSO to obtain the Pareto optimal points as well as the shape of the Pareto front.
2. Applying the weighted sum approach with fixed or adaptive weights.
3. Adopting the well-known GA approach VEGA for MCDM problems to the PSO framework to develop multi-swarm PSO to be implemented in MCDM problems in an effective manner.

The study can be considered the corner stone of applying PSO to solve such MCDM problems. Salman et al. (2002) proposed a PSO to task assignment. The PSO system combines local search methods (through self-experience) with global search methods (through neighboring experience), attempting to balance exploration and exploitation. A scan of some international electronic databases indicated that PSO has not applied yet in solving FMCDM problems.

1.5 Conclusions

From the above analysis, one can conclude that the implementation of AI techniques to handle FMCDM problems has occupied a reasonable amount of attention from the researchers with respect to some AI techniques such as ES, ANN, and GAs. But other techniques have not been opened yet such as SA, TS, PSO, DNA, and parallel hybrid techniques for handling the problems of FMCDM. However, the AI techniques that have been applied proved that they have the following advantages when dealing with FMCDM problems:

1. They have the possibility to consider the qualitative factors in the model structure and the solution procedure.
2. They can handle the decision maker's preferences, which are characterized as fuzzy preferences.
3. They can deal with a large amount of data that can be used in solving FMCDM problems.
4. The availability to estimate the aspiration levels in FMCDM.
5. The ability to estimate (determine) the membership functions that can be implemented to transform the FMCDM problem into a crisp problem to be handled easily.
6. The possibility to search and scan the search space in fuzzy multi-criteria combinatorial optimization problems where the search space is very large.
7. The AI techniques successes in solving different real-life problems such as scheduling, manufacturing, chemical, managerial, and other industrial applications.

1.5.1 Research Directions

The future research direction in this area is viewed from two angles:

1. Improving the performance of intelligent techniques by combining two or more of these techniques to get more powerful ones.
2. Implementing the available techniques to handle the FMCDM problems.

We shall talk about each individual case.

First: Improving the available techniques:

a) The mathematical background of these techniques needs more investigation and analysis.
b) Extending the AI techniques to handle more problems regarding FMCDM.
c) Studying the possibility and validity of combining more than two of these techniques to outperform the original ones.
d) Developing a comparative study between the AI techniques (metaheuristic techniques) to measure the performance of each one with respect to others. On the other side, measuring the performance and/or the quality of the solution(s) when changing the parameters of each technique.
e) Lights should be placed on new hybrid techniques as well as on parallel hybrid techniques that will be probably perform better than the AI techniques themselves.

Second: Intelligent FMCDM research directions:

This area of research still needs intensive research such as the following directions:

a) Large-scale FMCDM with mixed integer decision variables needs more investigation especially by using parallel hybrid intelligent systems to reduce the solution time.
b) Measuring the performance of AI techniques in higher dimensional FMCDM problems where the only test of performance is using benchmark functions. In addition, the theoretical analysis of measuring AI performance needs a look from the researchers.
c) Developing the theoretical analysis to deal with the FMCDM problems in its fuzzy environment without transforming it into crisp model, where the resulting solution may be more reasonable than the solution results from the transformation process.
d) Studying the effect of changing the AI techniques parameters on the solution behavior of FMCDM problems. In other words, understanding

the dynamics of swarm's dynamics (as in PSO) and the Pheromones dynamics (as in ACO) on the behavior of the optimization process.

e) Until now, no one has tried to open the area on doing both parametric and sensitivity analysis of MCDM and/or FMCDM by applying the AI techniques. The time is suitable for performing intelligent parametric analysis of MCDM and/or FMCDM problems. The results may be better than the traditional techniques for both linear and nonlinear FMCDM problems. As an idea, conduct the study of intelligent parametric analysis based on satisfying Kuhn−Tucker conditions or look for another easy way to do that.

f) Developing an intelligent system that combined most AI techniques to deal with FMCDM problems. For example, ES, ANN, SA, GA, and PSO may be combined in the following manner:
   - ES may handle the fuzzy preferences and other qualitative factors that have a great impact on the FMCDM problem behavior. This phase can be used as an evaluation process of the result solution(s).
   - Applying GA as a second phase to scan the solution space to get a satisfactory Pareto optimal solution.
   - Improving the performance of a PSO-based ANN with SA to use the GA output as an initial solution to this phase as a trial to obtain a better solution than the one in step (b).

This is a proposed scenario, and the researchers can change this scenario in different manners. More attention can be paid to measure the performance, and effectiveness should be done to compare the results with the existing techniques.

1. The ANN (for example) can be used to generate a reasonable membership function for solving the FMCDM problems based on the desires of the DM and/or the historical data of the problem.
2. Applying the AI techniques to implement the ranking approaches to deal with FMCDM problems.
3. Developing new approaches based on AI techniques to handle the fuzzy multi-attribute decision-making problems where a little research work has been done in this area.
4. Implementing AI techniques to solve FMCDM in the presence of multiple decision makers with indifference preferences information.
5. Invoking AI techniques in both interactive and goal programming to solve FMCDM. For example, developing an ANN to capture and represent the decision maker's preferences to support the search process for obtaining the most desirable solution.

6. The hybridization of fuzzy logic and evolutionary computation in what is called genetic fuzzy systems became an important research area during the last decade, and the results should be applied to deal with FMCDM to solve the problem without transforming it into a crisp model.

Last but not least, the implementation of AI techniques to solve the different problems of both FMCDM and MCDM will occupy a wide range of research in the next 20 years because of their ability to handle many complicated problems.

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FUZZY ANALYTIC HIERARCHY PROCESS AND ITS APPLICATION

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Abstract: The analytic hierarchy process (AHP) is one of the most widely-used multi-attribute decision-making methods. In this section we overview the fuzzy AHP methods existing in the literature. We present the four different approaches of fuzzy AHP methods by giving numerical examples.

Key words: Multi-attribute decision-making, fuzzy AHP, extent analysis, entropy value

1. INTRODUCTION

The analytic hierarchy process (AHP) is one of the most widely-used multi-attribute decision-making (MADM) methods. In any planning and decision-making process, a systematic and logical approach is used to arrive at the solution. In the multi-criteria decision analysis, the fuzzy set theory might be the most common method in dealing with uncertainty.

The analytic hierarchy process has been used in many different fields as a multi-attribute decision analysis tool with multiple alternatives and criteria. AHP uses “pair-wise comparisons” and matrix algebra to weight criteria. The decision is made by using the derived weights of the evaluative criteria (Saaty, 1980).

Importance is measured on an integer-valued 1−9 scale, with each number having the interpretation shown in Table 1.

C. Kahraman (ed.), Fuzzy Multi-Criteria Decision Making.

In this chapter, we give the literature review results in the following section. Section 2.1 presents an introduction and a definition of fuzzy AHP. Sections 2.2, 2.4, 2.6, and 2.9 present Van Laarhoven and Pedrycz's approach, Buckley's fuzzy AHP, Chang's extent analysis method, and fuzzy AHP with entropy value, with numerical examples, respectively. The last section summarizes suggestions for additonal research.

Table 1. Interpretation of Entities in a Pair-wise Comparison Matrix

Value of a_{ij}	**Interpretation**
1	Objectives *i* and *j* have equal importance
3	Objective *i* is weakly more important than objective *j*
5	Experience and judgment indicate that objective *i* is strongly more important than objective *j*
7	Objective *i* is very strongly or demonstrably more important than objective *j*
9	Objective *i* is absolutely more important than objective *j*
2, 4, 6, 8	Intermediate values

2. LITERATURE REVIEW

Many fuzzy AHP methods are proposed by various authors. These methods are systematic approaches to the alternative selection and justification problem by using the concepts of fuzzy set theory (Zadeh, 1965) and hierarchical structure analysis. Decision makers usually find that it is more confident to give interval judgments than fixed value judgments. Because usually he/she cannot be explicit about his/her preferences because of the fuzzy nature of the comparison process.

The earliest work in fuzzy AHP appeared in van Laarhoven and Pedrycz (1983), which compared fuzzy ratios described by triangular membership functions. Buckley (1985) determines fuzzy priorities of comparison ratios membership functions trapezoidal. Stam et al. (1996) explore how recently developed artificial intelligence techniques can be used to determine or approximate the preference ratings in AHP. They conclude that the feed-forward neural network formulation appears to be a powerful tool for analyzing discrete alternative multi-criteria decision problems with imprecise or fuzzy ratio-scale preference judgments. Chang (1996) introduces a new approach for handling fuzzy AHP, with the use of triangular fuzzy numbers for pair-wise comparison scale off fuzzy AHP and the use of the extent analysis method for the synthetic extent values of the pair-wise comparisons. Cheng (1997) proposes a new algorithm for evaluating naval tactical missile systems by the fuzzy analytical hierarchy process based on grade value of membership function. Weck et al. (1997) present a method to evaluate different production cycle alternatives adding the mathematics off fuzzy logic to the classic AHP. Any production cycle evaluated in this manner yields a fuzzy set. The outcome of the analysis can finally be defuzzified by forming the surface center of gravity of any fuzzy set, and the alternative production cycles investigated can be ranked in terms of the main objective set. Kahraman et al. (1998) use a fuzzy objective and subjective method obtaining the weights from AHP and make a fuzzy weighted evaluation. Cheng et al. (1999) propose a new method for evaluating weapon systems by analytical hierarchy process based on linguistic variable weight. Zhu et al. (1999) make a discussion on extent analysis method and applications of fuzzy AHP. Badri (2001) proposed a combined AHP-GP model for quality control systems. Creed (2001), Jansen et al. (2001) and Martinez-Tome et al. (2000) investigate food industry, customer satisfaction and food supply chain. Cebeci (2001) and Cebeci and Kahraman (2002) proposed a fuzzy AHP model to Measure customer satisfaction of catering service companies. Yu (2002) incorporates an absolute term linearization technique and a fuzzy rating expression into a GP-AHP model for solving group decision-making fuzzy AHP problem. Kahraman et al. (2004) provide an analytical tool to select the best Turkish catering firm providing the most customer satisfaction. The fuzzy analytic hierarchy process is used to compare three Turkish catering firms in their paper. Tolga et al. (2005) aim at creating an operating system selection framework for decision makers. Since decision makers have to consider both economic and noneconomic aspects of technology selection, both factors are considered in their developed framework. They develop the economic part of the decision process by fuzzy replacement analysis. Noneconomic factors and financial figures are combined using a fuzzy analytic hierarchy process approach. Hsiao and Chou (2006) propose a gestalt-like perceptual measure method by combining gestalt grouping principles and fuzzy entropy. The purpose of the proposed method is not to evaluate the grades of alternatives but to measure the gestalt-like perceptual degrees for home page design. They identify the weights using fuzzy AHP.

2.1 Fuzzy AHP

Inability of AHP to deal with the imprecision and subjectiveness in the pair-wise comparison process has been improved in fuzzy AHP. Instead of a crisp value, fuzzy AHP uses a range of value to incorporate the decision maker's uncertainty (Kuswandari, 2004).

2.2 Van Laarhoven and Pedrycz's Approach (1983)

Van Laarhoven and Pedrycz (1983) offer an algorithm that is the direct extension of Saaty's AHP method. They identify the weights through the AHP operations. In that study, Laarhoven and Pedrycz use the triangular fuzzy numbers. The computation steps are the same as those in crisp AHP. The Lootsma's logarithmic least-squares method is used to derive fuzzy weights and fuzzy performance scores (Chen et al., 1992).

Laarhoven and Pedrycz's approach is shown by the following steps:

Step 1. Consult with the MCDMs and obtain n+1 fuzzy reciprocal matrix that takes the following form as shown (1).

$$\tilde{A} = \begin{bmatrix} (1,1,1) & \begin{matrix} \tilde{a}_{121} \\ \tilde{a}_{122} \\ \vdots \\ \tilde{a}_{12P_{12}} \end{matrix} & \cdots & \begin{matrix} \tilde{a}_{1n1} \\ \tilde{a}_{1n2} \\ \vdots \\ \tilde{a}_{1nP_{1n}} \end{matrix} \\ \begin{matrix} \tilde{a}_{211} \\ \tilde{a}_{212} \\ \vdots \\ \tilde{a}_{21P_{21}} \end{matrix} & (1,1,1) & \cdots & \begin{matrix} \tilde{a}_{2n1} \\ \tilde{a}_{2n2} \\ \vdots \\ \tilde{a}_{2nP_{2n}} \end{matrix} \\ \vdots & \vdots & \ddots & \vdots \\ \begin{matrix} \tilde{a}_{n11} \\ \tilde{a}_{n12} \\ \vdots \\ \tilde{a}_{n1P_{n1}} \end{matrix} & \begin{matrix} \tilde{a}_{n21} \\ \tilde{a}_{n22} \\ \vdots \\ \tilde{a}_{n2P_{n2}} \end{matrix} & \cdots & (1,1,1) \end{bmatrix} \quad (1)$$

where $\tilde{a}_{ijP_{ij}}$ are fuzzy ratios estimated by multiple decision makers. Note that p_{ij} may be 0 when no decision maker expresses his/her comparison ratios or greater than 1 when more than one decision maker expresses his/her comparison ratios.

Step 2. Let $z_i = \left(l_i, m_i, u_i\right)$. Solve the following linear equations:

$$l_i \left(\sum_{\substack{j=1 \\ j \neq i}}^{n} P_{ij} \right) - \sum_{\substack{j=1 \\ j \neq 1}}^{n} P_{ij} u_j = \sum_{\substack{j=1 \\ j \neq 1}}^{n} \sum_{k=1}^{P_{ij}} \left(\ln l_{ijk}\right), \quad \forall \mathrm{i} \tag{2}$$

$$m_i \left(\sum_{\substack{j=1 \\ j \neq i}}^{n} P_{ij} \right) - \sum_{\substack{j=1 \\ j \neq i}}^{n} P_{ij} m_j = \sum_{\substack{j=1 \\ j \neq i}}^{n} \sum^{P_{ij}} \left(\ln m_{ijk}\right), \quad \forall \mathrm{i} \tag{3}$$

$$u_i \left(\sum_{\substack{j=1 \\ j \neq i}}^{n} P_{ij} \right) - \sum_{\substack{j=1 \\ j \neq i}}^{n} P_{ij} l_j = \sum_{\substack{j=1 \\ j \neq i}}^{n} \sum^{P_{ij}} \left(\ln u_{ijk}\right), \quad \forall \mathrm{i} \tag{4}$$

As $\ln\left(l_{ijk}\right)$ and $\ln\left(u_{ijk}\right)$ are lower and upper values of $\ln\left(a_{ijk}\right) = -\ln\left(a_{jik}\right)$, the following must hold true [see Eq. (2)]:

$$\ln\left(l_{ijk}\right) + \ln\left(l_{jik}\right) = \ln\left(u_{ijk}\right) + \ln\left(u_{jik}\right) = 0, \quad \forall \mathrm{i, j, k.} \tag{5}$$

Thus Eqs. (2) and (4) are linear dependent. The same holds for Eq. (3). Generally, a solution for Eqs. (2), (3), and (4) is given as:

$$z_i = \left(l_i + t_1,\ m_i + t_2,\ u_i + t_1\right), \quad \forall \mathrm{i} \tag{6}$$

where t_1 and t_2 can be chosen arbitrarily.

Step 3. The right sides of the equations above are operated using logarithmic operations. Then we obtain the fuzzy weight in Eq. (7):

$$w_i = \left(\lambda_1 \exp\left(l_i\right), \lambda_2 \exp\left(m_i\right), \lambda_3 \exp\left(u_i\right)\right) \tag{7}$$

where

$$\lambda_1 = \left[\sum_{i=1}^{n} \exp\left(u_i\right)\right]^{-1} \quad \lambda_2 = \left[\sum_{i=1}^{n} \exp\left(m_i\right)\right]^{-1} \quad \lambda_3 = \left[\sum_{i=1}^{n} \exp\left(l_i\right)\right]^{-1}$$

Equation (7) can also be used to determine the performance score r_{ij} .

Step 4. Steps 1−3 are repeated several times until all reciprocal matrices are solved. With the fuzzy weights and performance scores, we can calculate the fuzzy utility for alternative A_i as

$$u_i = \sum_{j=1}^{n} w_j r_{ij} \tag{8}$$

2.3 A Numerical Example

A company is looking for a sales manager. There are four applicants for this position. The company is also looking for four attributes from these applicants. These attributes are *leadership*, *mathematic creativity*, *communication skill*, and *experimentation*. Figure 1 shows the hierarchy of sales manager selection problem. Three decision makers will be graded for the four attributes.

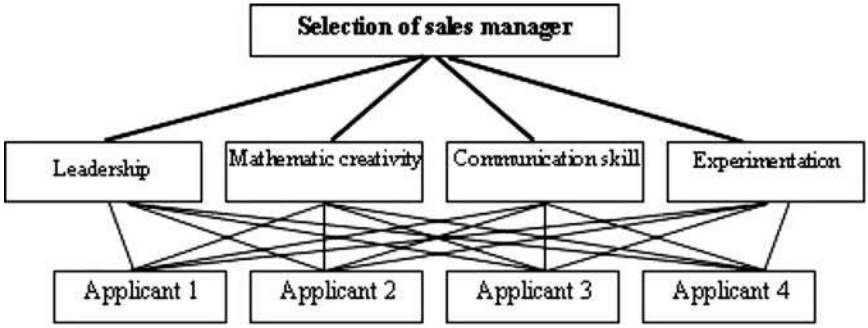

Figure 1. The hierarchy of the sales manager selection

The three decision makers' opinions about the relative importance of a pair of attributes are shown in Tables 2 to 6.

Table 2. Pair-Wise Comparisons of Applicants for Leadership

	A_1	A_2	A_3	A_4
A_1	(1, 1, 1)	(2/5, 1/2, 2/3) (2/5, 1/2, 2/3)	(2/3, 1, 3/2)	(2/9, 1/4, 2/7) (2/9, 1/4, 2/7) (2/7, 1/3, 2/5)
A_2	(3/2, 2, 5/2) (3/2, 2, 5/2)	(1, 1, 1)	(5/2, 3, 7/2) (3/2, 2, 5/2)	(2/5, 1/2, 2/3) (2/3, 1, 3/2) (2/3, 1, 3/2)
A_3	(2/3, 1, 3/2)	(2/7, 1/3, 2/5) (2/5, 1/2, 2/3)	(1, 1, 1)	(2/5, 1/2, 2/3) (2/3, 1, 3/2)
A_4	(7/2, 4, 9/2) (7/2, 4, 9/2) (5/2, 3, 7/2)	(3/2, 2, 5/2) (2/3, 1, 3/2) (2/3, 1, 3/2)	(3/2, 2, 5/2) (2/3, 1, 3/2)	(1, 1, 1)

Table 3. Pair-Wise Comparisons of Applicants for Mathematic Creativity

	A_1	A_2	A_3	A_4
A_1	(1, 1, 1)	(3/2, 2, 5/2) (3/2, 2, 5/2)	(2/7, 1/3, 2/5) (2/9, 1/4, 2/7) (2/9, 1/4, 2/7)	(2/3, 1, 3/2)
A_2	(2/5, 1/2, 2/3) (2/5, 1/2, 2/3)	(1, 1, 1)	(2/5, 1/2, 2/3)	(2/5, 1/2, 2/3) (2/3, 1, 3/2)
A_3	(5/2, 3, 7/2) (7/2, 4, 9/2) (7/2, 4, 9/2)	(3/2, 2, 5/2)	(1, 1, 1)	(5/2, 3, 7/2) (5/2, 3, 7/2) (3/2, 2, 5/2)
A_4	(2/3, 1, 3/2)	(3/2, 2, 5/2) (2/3, 1, 3/2)	(2/7, 1/3, 2/5) (2/7, 1/3, 2/5) (2/5, 1/2, 2/3)	(1, 1, 1)

Table 4. Pair-Wise Comparisons of Applicants for Communication Skill

	A_1	A_2	A_3	A_4
A_1	(1, 1, 1)	(5/2, 3, 7/2) (5/2, 3, 7/2)	(5/2, 3, 7/2)	(2/3, 1, 3/2)
A_2	(2/7, 1/3, 2/5) (2/7, 1/3, 2/5)	(1, 1, 1)	(2/3, 1, 3/2)	(3/2, 2, 5/2) (3/2, 2, 5/2)
A_3	(2/7, 1/3, 2/5)	(2/3, 1, 3/2)	(1, 1, 1)	(2/7, 1/3, 2/5)
A_4	(2/3, 1, 3/2)	(2/5, 1/2, 2/3) (2/5, 1/2, 2/3)	(5/2, 3, 7/2)	(1, 1, 1)

Table 5. Pair-Wise Comparisons of Applicants for Experimentation

	A_1	A_2	A_3	A_4
A_1	(1, 1, 1)	(2/3, 1, 3/2)	(7/2, 4, 9/2) (5/2, 3, 7/2)	(2/5, 1/2, 2/3)
A_2	(2/3, 1, 3/2)	(1, 1, 1)	(3/2, 2, 5/2) (5/2, 3, 7/2) (5/2, 3, 7/2)	(2/3, 1, 3/2)
A_3	(2/9, 1/4, 2/7) (2/7, 1/3, 2/5)	(2/5, 1/2, 2/3) (2/7, 1/3, 2/5) (2/7, 1/3, 2/5)	(1, 1, 1)	(2/7, 1/3, 2/5)
A_4	(3/2, 2, 5/2)	(2/3, 1, 3/2)	(5/2, 3, 7/2)	(1, 1, 1)

Table 6. Pair-Wise Comparisons of Attributes

	X_1	X_2	X_3	X_4
X_1	(1, 1, 1)	(3/2, 2, 5/2) (3/2, 2, 5/2) (2/3, 1, 3/2)	(7/2, 4, 9/2)	(2/3, 1, 3/2)
X_2	(2/5, 1/2, 2/3) (2/5, 1/2, 2/3) (2/3, 1, 3/2)	(1, 1, 1)	(3/2, 2, 5/2)	(2/5, 1/2, 2/3) (2/3, 1, 3/2)
X_3	(2/9, 1/4, 2/7)	(2/5, 1/2, 2/3)	(1, 1, 1)	(2/9, 1/4, 2/7)
X_4	(2/3, 1, 3/2)	(3/2, 2, 5/2) (2/3, 1, 3/2)	(7/2, 4, 9/2)	(1, 1, 1)

The following phases are taken to solve the problem:

$$l_1\left(\sum_{j=2}^{4} P_{1j}\right) - \sum_{j=2}^{4} P_{1j}u_j = \sum_{j=2}^{4}\sum_{k=1}^{P_{ij}} \ln\left(l_{1jk}\right)$$

$$l_1\left(P_{12}+P_{13}+P_{14}\right)-\left(P_{12}u_2+P_{13}u_3+P_{14}u_4\right)=\sum_{k=1}^{P_{12}}\ln\left(l_{12k}\right)+\sum_{k=1}^{P_{13}}\ln\left(l_{13k}\right)+\sum_{k=1}^{P_{14}}\ln\left(l_{14k}\right)$$

where

$P_{12} = 2$ (two decision makers)
$P_{13} = 1$ (one decision maker)
$P_{14} = 3$ (three decision makers)

$$6l_1 - 2u_2 - u_3 - 3u_4 = \begin{pmatrix} \ln(2/5) + \ln(2/5) + \ln(2/3) \\ + \ln(2/9) + \ln(2/9) + \ln(2/7) \end{pmatrix}$$

$$l_2 \left(\sum_{\substack{j=1 \\ j \neq 2}}^{4} P_{2j} \right) - \sum_{\substack{j=1 \\ j \neq 2}}^{4} P_{2j} u_j = \sum_{\substack{j=1 \\ j \neq 2}}^{4} \sum_{k=1}^{P_{ij}} \ln(l_{2jk})$$

$$l_2 (P_{21} + P_{23} + P_{24}) - (P_{21}u_1 + P_{23}u_3 + P_{24}u_4) =$$
$$\sum_{k=1}^{P_{21}} \ln(l_{21k}) + \sum_{k=1}^{P_{23}} \ln(l_{23k}) + \sum_{k=1}^{P_{24}} \ln(l_{24k})$$

where

$P_{21} = 2$ (two decision makers)
$P_{23} = 2$ (two decision makers)
$P_{24} = 3$ (three decision makers)

$$7l_2 - 2u_1 - 2u_3 - 3u_4 = \begin{pmatrix} (\ln(3/2) + \ln(3/2)) + (\ln(5/2) + \ln(3/2)) \\ + (\ln(2/5) + \ln(2/3) + \ln(2/3)) \end{pmatrix}$$

By a similar process, obtained linear equations can be represented as

$$\begin{aligned}
6l_1 - 2u_2 - 1u_3 - 3u_4 &= -6.4989 \\
7l_2 - 2u_1 - 2u_3 - 3u_4 &= 0.4054 \\
5l_3 - 1u_1 - 2u_2 - 2u_4 &= -3.8962 \\
8l_4 - 3u_1 - 3u_2 - 2u_3 &= 3.0163 \\
6m_1 - 2m_2 - 1m_3 - 3m_4 &= -5.2574 \\
7m_2 - 2m_1 - 2m_3 - 3m_4 &= 0.4054 \\
5m_3 - 1m_1 - 2m_2 - 2m_4 &= -3.8962 \\
8m_4 - 3m_1 - 3m_2 - 2m_3 &= 3.0163 \\
6u_1 - 2l_2 - 1l_3 - 3l_4 &= -3.8272 \\
7u_2 - 2l_1 - 2l_3 - 3l_4 &= 4.4071 \\
5u_3 - 1l_1 - 2l_2 - 2l_4 &= -0.9162 \\
8u_4 - 3l_1 - 3l_2 - 2l_3 &= 7.3098
\end{aligned}$$

The solutions to these equations are given in Table 7:

Table 7. The Solutions to the Equations

I	l_i	m_i	u_i
1	0	0	0.1443
2	0.7870	0.8919	1.1028
3	0.1936	0.2849	0.5216
4	0.9751	1.0629	1.2572

The exponentials of l_i, m_i, and u_i are given in Table 8:

Table 8. The Exponentials of l_i, m_i, and u_i

I	$\exp(l_i)$	$\exp(m_i)$	$\exp(u_i)$
1	1.0000	1.0000	1.1552
2	2.1967	2.4397	3.0125
3	1.2136	1.3296	1.6847
4	2.6514	2.8947	3.5155

We can calculate the fuzzy performance score r_{11} using Eq. (7) with the exponential numbers.

$$r_{11} = (\lambda_1 \exp(l_1), \lambda_2 \exp(m_1), \lambda_3 \exp(u_1))$$

The terms λ_1, λ_2, and λ_3 are computed as

$$\lambda_1 = \left(\sum_{i=1}^{4} \exp(u_i)^{-1} \right) = 0.1067 \;, \lambda_2 = \left(\sum_{i=1}^{4} \exp(m_i)^{-1} \right) = 0.1304$$

$$\lambda_3 = \left(\sum_{i=1}^{4} \exp(l_i)^{-1} \right) = 0.1416 \,.$$

The fuzzy performance scores r_{1j} , j:1,2,3,4, can be summarized as

$$r_{11} = (0.1076, 0.1304, 0.1635)$$
$$r_{12} = (0.2343, 0.3181, 0.4265)$$
$$r_{13} = (0.1294, 0.1733, 0.2385)$$
$$r_{14} = (0.2829, 0.3774, 0.4977).$$

Steps 1 through 3 are applied to Tables 3, 4, 5, and 6. All results are given in Table 9.

Table 9. All Results

	X1	X2	X3	X4
A1	(0.1067, 0.1304, 0.1635)	(0.1434, 0.1793, 0.2225)	(0.3498, 0.4363, 0.5362)	(0.2018, 0.2729, 0.3730)
A2	(0.2343, 0.3181, 0.4265)	(0.1035, 0.1288, 0.1680)	(0.1708, 0.2190, 0.2787)	(0.1868, 0.2760, 0.4037)
A3	(0.1295, 0.1733, 0.2385)	(0.4457, 0.5049, 0.5568)	(0.1042, 0.1313, 0.1685)	(0.0855, 0.0974, 0.1139)
A4	(0.2829, 0.3774, 0.4977)	(0.1413, 0.1868, 0.2495)	(0.1641, 0.2130, 0.2827)	(0.2556, 0.3530, 0.4772)

W = [(0.2579, 0.3509, 0.4703), (0.1609, 0.2199, 0.3054), (0.0812, 0.0932, 0.1101), (0.2418, 0.3354, 0.4612)]

We can calculate fuzzy utilities U_1, U_2, U_3, and U_4 by Eq. (8) as:

$$U_1 = (0.1277, 0.2173, 0.3759)$$
$$U_2 = (0.1361, 0.2529, 0.4687)$$
$$U_3 = (0.1342, 0.2167, 0.3532)$$
$$U_4 = (0.1708, 0.3117, 0.5614).$$

The fuzzy utilities can be ranked by any appropriate fuzzy ranking method.

2.4 Buckley's (1985) Fuzzy AHP

Buckley also extended Saaty's AHP method to incorporate fuzzy comparison ratios a_{ij}. He pointed out that Van Laarhoven and Pedrycz's (1983) method was subject to two problems. First, the linear equations of obtained equations do not always have a unique solution. Second, they insist on obtaining triangular fuzzy numbers for their weights.

Buckley's (1985) approach is shown in the following steps.

Step 1. Consult the decision maker, and obtain the comparison matrix A whose elements are $\tilde{t}_{ij} = (a_{ij}, b_{ij}, c_{ij}, d_{ij})$, where all i and j are trapezoidal fuzzy numbers.

Step 2. The fuzzy weights w_i can be calculated as follows. The geometric mean for each row is determined as

$$\widetilde{z}_i = \left[\prod_{j=1}^{n} \widetilde{t}_{ij} \right]^{1/n}, \text{ for all } i \tag{9}$$

The fuzzy weight w_i is given as

$$w_i = \widetilde{z}_i \oplus \left[\sum_{j=1}^{n} \widetilde{z}_j \right]^{-1}. \tag{10}$$

In the following discussion, we will detail the derivation of fuzzy weight w_i . Let the left leg and right leg of $\widetilde{t}_{ij}$ be, respectively, defined as

$$f_i(\alpha) = \left[\prod_{j=1}^{n} \left(\left(b_{ij} - a_{ij} \right) \alpha + a_{ij} \right) \right]^{1/n}, \quad \alpha \in [0,1] \tag{11}$$

$$g_i(\alpha) = \left[\prod_{j=1}^{n} \left(\left(c_{ij} - d_{ij} \right) \alpha + b_{ij} \right) \right]^{1/n}, \quad \alpha \in [0,1]. \tag{12}$$

Furthermore, let

$$a_i = \left[\prod_{j=1}^{n} \widetilde{t}_{ij} \right]^{1/n} \tag{13}$$

and

$$a = \sum_{i=1}^{m} a_i \ . \tag{14}$$

Similarly, we can define b_i and b, c_i and c, and d_i and d. The fuzzy weight w_i is determined as

$$w_i = \left(\frac{a_i}{a}, \frac{b_i}{b}, \frac{c_i}{c}, \frac{d_i}{d} \right), \ \forall \text{i} \tag{15}$$

where the membership function $\mu_{wi}(x)$ is defined as follows: Let x be a real number on the horizontal axis. The $\mu_{wi}(x)$ can be summarized as in Table 10.

Table 10. Interpretation of Entities in a Pair-wise Comparison Matrix

X	$\mu_{w_i}(x)$
$\leq (a_i/d)$	0
$\geq (a_i/d)$	0
$\left[\frac{b_i}{c}, \frac{c_i}{b}\right]$	1
$\left[\frac{a_i}{d}, \frac{b_i}{c}\right]$	$\alpha \in [0,1]$
$\left[\frac{c_i}{b}, \frac{d_i}{a}\right]$	$\alpha \in [0,1]$

When $x \in \left[\frac{a_i}{d}, \frac{b_i}{c}\right]$ or $x \in \left[\frac{c_i}{b}, \frac{d_i}{a}\right]$, the x is calculated as

$$x = \begin{cases} f_i(\alpha)/g(\alpha), & \text{if } x \in [a_i/d,\ b_\text{i}/c] \\ g_i(\alpha)/f(\alpha), & \text{if } x \in [c_i/b,\ d_\text{i}/a] \end{cases} \quad (16)$$

where $f(\alpha) = \sum_{i=1}^{m} f_i(\alpha)$ and $g(\alpha) = \sum_{i=1}^{m} g_i(\alpha)$,

Step 2 is repeated for all the fuzzy performance scores.

Step 3. The fuzzy weights and fuzzy performance scores are aggregated. The fuzzy utilities U_i, $\forall$i, are obtained based on

$$U_i = \sum_{j=1}^{n} w_j r_{ij}, \quad \forall i. \quad (17)$$

2.5 A Numerical Example

A ceramic factory is looking for a general manager. There are three applicants for this position. The company is also looking for four attributes from these applicants. These attributes are *leadership*, *problem-solving skill*, *communication skill*, and *experimentation*. An expert will be graded for the four attributes. The expert opinions about the relative importance of a pair of attributes are shown in Tables 11 to 15.

Table 11. Pair-Wise Comparison of Applicants for Leadership

	A1	A2	A3
A1	(1, 1, 1, 1)	(1, 2, 2, 3)	(2, 2, 4, 4)
A2	(1/3, 1/2, 1/2, 1)	(1, 1, 1, 1)	(1, 2, 2, 3)
A3	(1/4, 1/4, 1/2, 1/2)	(1/3, 1/2, 1/2, 1)	(1, 1, 1, 1)

Table 12. Pair-Wise Comparison of Applicants for Leadership

	A1	A2	A3
A1	(1, 1, 1, 1)	(1/4, 1/3, 1/3, 1/2)	(1, 1, 2, 2)
A2	(2, 3, 3, 4)	(1, 1, 1, 1)	(3, 3, 4, 4)
A3	(1/2, 1/2, 1, 1)	(1/4, 1/4, 1/3, 1/3)	(1, 1, 1, 1)

Table 13. Pair-Wise Comparison of Applicants for Leadership

	A1	A2	A3
A1	(1, 1, 1, 1)	(6, 6, 7, 7)	(3, 3, 4, 4)
A2	(1/7, 1/7, 1/6, 1/6)	(1, 1, 1, 1)	(1/2, 1/2, 1, 1)
A3	(1/4, 1/4, 1/3, 1/3)	(1, 1, 2, 2)	(1, 1, 1, 1)

Table 14. Pair-Wise Comparison of Applicants for Leadership

	A1	A2	A3
A1	(1, 1, 1, 1)	(1/7, 1/6, 1/6, 1/5)	(1, 1, 2, 2)
A2	(5, 6, 6, 7)	(1, 1, 1, 1)	(1, 2, 2, 3)
A3	(1/2, 1/2, 1, 1)	(1/4, 1/4, 1/3, 1/3)	(1, 1, 1, 1)

Table 15. Pair-Wise Comparison of Attributes

	X1	X2	X3	X4
X1	(1, 1, 1, 1)	(1, 2, 2, 3)	(2, 2, 3, 3)	(1/3, 1/3, 1/3, 1/3)
X2	(1/3, 1/2, 1/2, 1)	(1, 1, 1, 1)	(1, 1, 2, 2)	(1/2, 1/3, 1/3, 1/2)
X3	(1/3, 1/3, 1/2, 1/2)	(1/2, 1/2, 1, 1)	(1, 1, 1, 1)	(1/2, 1/2, 1/2, 1/2)
X4	(3, 3, 3, 3)	(2, 3, 3, 4)	(2, 2, 2, 2)	(1, 1, 1, 1)

The following phases are taken to solve the problem: For the first reciprocal matrix, the geometric mean is

$$a_1 = \left(\prod_{j=1}^{3} a_{1j} \right)^{1/3} = (a_{11} \times a_{12} \times a_{13}) = (1 \times 1 \times 2)^{1/3} = 1.2599$$

$$a_2 = \left(\prod_{j=1}^{3} a_{2j} \right)^{1/3} = (a_{21} \times a_{22} \times a_{23}) = (1/3 \times 1 \times 1)^{1/3} = 0.6933$$

$$a_3 = \left(\prod_{j=1}^{3} a_{3j} \right)^{1/3} = (a_{31} \times a_{32} \times a_{33}) = (1/4 \times 1/3 \times 1)^{1/3} = 0.4367$$

Hence, $a = \sum_{i=1}^{3} a_i = 1.2599 + 0.6933 + 0.4367 = 2.3899$.

Similarly, we can get b_i and b, c_i and c, and d_i and d. They are summarized as in Table 16.

Table 16. Geometric Means

I	1	2	3	Sum of the *k* th row
a_i	1.2599	0.6933	0.4367	$\Sigma a_i = 2.3899$
b_i	1.5874	1	0.5	$\Sigma b_i = 3.0874$
c_i	2	1	0.6299	$\Sigma c_i = 3.6299$
d_i	2.2894	1.4422	0.7937	$\Sigma d_i = 4.5253$

Thus, $(a, b, c, d) = (2.3899, 3.0874, 3.6299, 4.5253)$.

The performance scores r_{j1}, $\mathrm{j} = 1, 2, \text{and } 3$ can be obtained as

$$r_{11} = \left(\frac{a_1}{d}, \frac{b_1}{c}, \frac{c_1}{b}, \frac{d_1}{a} \right) = (0.2784, 0.4373, 0.6477, 0.9579)$$

$$r_{21} = \left(\frac{a_2}{d}, \frac{b_2}{c}, \frac{c_2}{b}, \frac{d_2}{a} \right) = (0.1532, 0.2754, 0.3238, 0.6034)$$

$$r_{31}=\left(\frac{a_3}{d},\frac{b_3}{c},\frac{c_3}{b},\frac{d_3}{a}\right)=(0.0965, 0.1377, 0.2040, 0.3321)$$

The performance scores r_{j2}, r_{j3}, and w_i can be obtained as

r_{12} = (0.1495, 0.1797, 0.2668, 0.3333)
r_{22} = (0.4312, 0.5393, 0.6994, 0.8550)
r_{32} = (0.1186, 0.1296, 0.2118, 0.2352)
r_{13} = (0.5875, 0.5875, 0.8283, 0.8283)
r_{23} = (0.0930, 0.0930, 0.1501, 0.1501)
r_{33} = (0.1412, 0.1412, 0.2383, 0.2383)
r_{14} = (0.1170, 0.1288, 0.1888, 0.2111)
r_{24} = (0.5521, 0.6136, 0.7857, 0.8703)
r_{34} = (0.1119, 0.1170, 0.1888, 0.1987)
w_1 = (0.1725, 0.2278, 0.2758, 0.3427)
w_2 = (0.1025, 0.1354, 0.1762, 0.2604)
w_3 = (0.1025, 0.1139, 0.1640, 0.1841)
w_4 = (0.3554, 0.4367, 0.4778, 0.5765)

$$w_1 * r_{11} = \{(a_1,\ a_2)[L_1,\ L_2],\ b_1b_2,\ c_1c_2,\ (d_1d_2)[R_1,\ R_2]\}$$

$r_{11} = (a_1, b_1, c_1, d_1)$, $w_1 = (a_2, b_2, c_2, d_2)$
$L_1 = (b_1-a_1)(b_2-a_2)$, $L_2 = a_2(b_1-a_1) + a_1(b_2-a_2)$
$R_1 = (d_1-c_1)(d_2-c_2)$, $R_2 = -[d_2(d_1-c_1) + d_1(d_2-c_2)]$

Table 17. The Values of $w_j r_{1j}$

J	$w_j r_{1j}$
1	{0.0480[0.00878, 0.0427], 0.0996, 0.1786, 0.3282[0.0207, −0.170]}
2	{0.0153[0.00099, 0.00799], 0.0243, 0.0470, 0.0883[0.0061, −0.0473]}
3	{0.0602[0, 0.0066], 0.0669, 0.1358, 0.1524[0, −0.0166]}
4	{0.0415, [0.00095, 0.0136], 0.0562, 0.0902, 0.1216[0.0022, −0.0336]}

U_1 = {0.165, [0.01072, 0.13119], 0.3159, 0.4516, 0.6905[0.029, −0.2678]}

According to Table 17, the membership function value of $\mu_{u1}(x)$ may be summarized as in Table 18.

Table 18. The Membership Function Value of $\mu_{u1}(x)$

X	$\mu_{u1}(x)$
≤ 0.165	0
≥ 0.6905	0
$0.3159 \le x \le 0.4516$	1
$0.165 \le x \le 0.3159$	$\alpha \in [0,1]$
$0.4516 \le x \le 0.6905$	$\alpha \in [0,1]$

When $x \in [0.165, 0.3159]$, it is defined as:

$x = (0.01072)\,\alpha^2 + (0.13119)\,\alpha + 0.165$

and when $x \in [0.4516, 0.6905]$, it is defined as:

$x = (0.029)\,\alpha^2 + (-0.2678)\,\alpha + 0.6905.$

The fuzzy utilities U_2, and U_3 can be obtained in a similar manner. They are also presented in Figure 2.

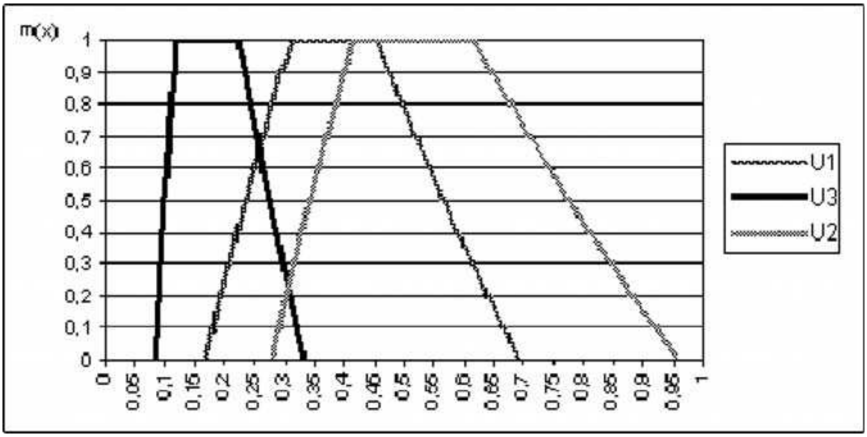

Figure 2. The fuzzy utilities

2.6 Chang's (1992) Extent Analysis Method

First, the outlines of the extent analysis method on fuzzy AHP are given and then the method is applied to a catering firm selection problem. Let $X = \{x_1, x_2, ..., x_n\}$ be an object set and $U = \{u_1, u_2, ..., u_m\}$ be a goal set. According to the method of Chang's (1992) extent analysis, each object is taken and extent analysis for each goal, g_i, is performed, respectively.

Therefore, m extent analysis values for each object can be obtained, with the following signs:

$$M_{gi}^{1}, M_{gi}^{2}, \ldots, M_{gi}^{m}, \qquad \mathrm{i} = 1,2,\ldots,\mathrm{n}$$

where all the $M_{gi}^{j}\left(j = 1,2,\ldots,m\right)$ are TFNs.

The steps of Chang's extent analysis can be given as in the following:

Step 1. The value of fuzzy synthetic extent with respect to *i*th object is defined as

$$S_i = \sum_{j=1}^{m} M_{gi}^{j} \otimes \left[\sum_{i=1}^{n} \sum_{j=1}^{m} M_{gi}^{j} \right]^{-1} \tag{18}$$

To obtain $\sum_{j=i}^{m} M_{gi}^{j}$, perform the fuzzy addition operation of m extent analysis values for a particular matrix such that

$$\sum_{j=1}^{m} M_{gi}^{j} = \left(\sum_{j=1}^{m} l_i, \sum_{j=1}^{m} m_i, \sum_{j=1}^{m} u_i \right) \tag{19}$$

and to obtain $\left[\sum_{i=1}^{n} \sum_{j=1}^{m} M_{gi}^{j}\right]^{-1}$, perform the fuzzy addition operation of $M_{gi}^{j}\left(j = 1,2,\ldots,m\right)$ values such that

$$\sum_{i=1}^{n} \sum_{j=1}^{m} = \left(\sum_{i=1}^{n} l_i, \sum_{i=1}^{n} m_i, \sum_{i=1}^{n} u_i \right) \tag{20}$$

and then compute the inverse of the vector in Eq. (20) such that

$$\left[\sum_{i=1}^{n} \sum_{j=1}^{m} M_{gi}^{j} \right]^{-1} = \left(\frac{1}{\sum_{i=1}^{n} u_i}, \frac{1}{\sum_{i=1}^{n} m_i}, \frac{1}{\sum_{i=1}^{n} l_i} \right) \tag{21}$$

Step 2. The degree of possibility of

$M_2 = (l_2, m_2, u_2) \geq M_1 = (l_1, m_1, u_1)$ is defined as

$$V(M_2 \geq M_1) = \sup_{y \geq x}[\min(\mu_{M1}(x), \mu_{M2}(y))] \quad (22)$$

and can be equivalently expressed as follows:

$$V(M_2 \geq M_1) = hgt(M_1 \cap M_2) = \mu_{M2}(d) = \begin{cases} 1, & \text{if } m_2 \geq m_1 \\ 0, & \text{if } l_1 \geq u_2 \\ \dfrac{l_1 - u_2}{(m_2 - u_2) - (m_1 - l_1)}, & \text{otherwise} \end{cases} \quad (23)$$

where d is the ordinate of highest intersection point D between μ_{M1} and μ_{M2} (see Figure 3).

To compare M_1 and M_2, we need both the values of $V(M_1 \geq M_2)$ and $V(M_2 \geq M_1)$.

Step 3. The degree of possibility for a convex fuzzy number to be greater than k convex fuzzy numbers $M_i = (i = 1, 2, ..., k)$ can be defined by

$$\begin{aligned} V(M \geq M_1, M_2, ..., M_k) &= V[(M \geq M_1)\, and\, (M \geq M_2) \text{ and ...and } (M \geq M_k)] \\ &= \min V(M \geq M_i), \quad \text{i} = 1, 2, ..., \text{k} \end{aligned} \quad (24)$$

Assume that

$$d'(A_i) = \min V(S_i \geq S_k) \quad (25)$$

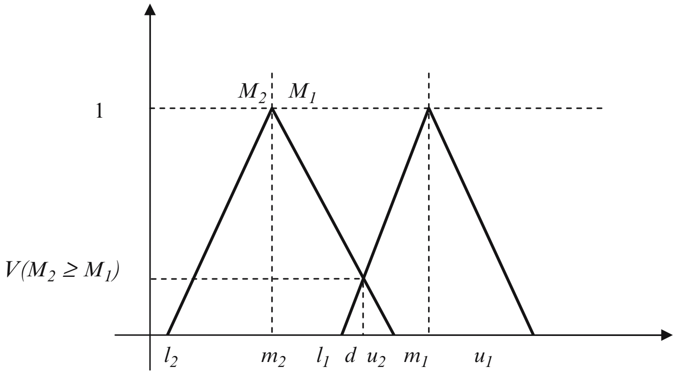

Figure 3. The intersection between M1 and M2

For $k = 1,2,...,n; k \neq i$. Then the weight vector is given by

$$W' = \left(d'(A_1), d'(A_2'), ..., d'(A_n)\right)^T \tag{26}$$

where $A_i\,(i = 1,2,...,n)$ are *n* elements.

Step 4. Via normalization, the normalized weight vectors are

$$W = \left(d(A_1), d(A_2), ..., d(A_n)\right)^T \tag{27}$$

where *W* is a nonfuzzy number.

In this method, the fuzzy conversion scale is as in Table 19. A different scale in fuzzy AHP can be found in the literature as in Abdel-Kader and Dugdale's (2001) study.

Table 19. Triangular Fuzzy Conversion Scale

Linguistic scale	Triangular fuzzy scale	Triangular fuzzy reciprocal scale
Just equal	(1, 1, 1)	(1, 1, 1)
Equally important	(1/2, 1, 3/2)	(2/3, 1, 2)
Weakly important	(1, 3/2, 2)	(1/2, 2/3, 1)
Strongly more important	(3/2, 2, 5/2)	(2/5, 1/2, 2/3)
Very strong more important	(2, 5/2, 3)	(1/3, 2/5, 1/2)
Absolutely more important	(5/2, 3, 7/2)	(2/7, 1/3, 2/5)

2.7 A Numerical Example (Kahraman et al., 2004)

A big company wants to contract with a catering firm. Alternative catering firms are Firm1, Firm2, and Firm3. The goal is to select the best catering among the alternatives. The selection hierarchy of the best catering firm is shown in Figure 4.

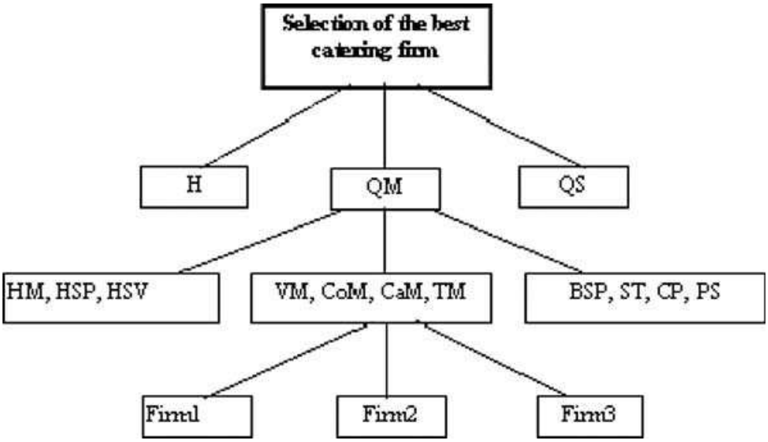

Figure 4. Selection of the best catering firm

The following phases are taken to solve the problem:
From Table 20,

$S_H = (3.17, 4.00, 5.00) \otimes (1/12.34, 1/10.00, 1/8.14) = (0.26, 0.40, 0.61)$

$S_{QM} = (2.90, 3.50, 4.17) \otimes (1/12.34, 1/10.00, 1/8.14) = (0.24, 0.35, 0.51)$

$S_{QS} = (2.07, 2.50, 3.17) \otimes (1/12.34, 1/10.00, 1/8.14) = (0.17, 0.21, 0.39)$ are obtained.

Table 20. The Fuzzy Evaluation Matrix with Respect to the Goal

	H	QM	QS
H	(1, 1, 1)	(3/2, 2, 5/2)	(2/3,1, 3/2)
QM	(2/5, 1/2, 2/3)	(1, 1, 1)	(3/2, 2, 5/2)
QS	(2/3, 1, 3/2)	(2/5, 1/2, 2/3)	(1, 1, 1)

Using these vectors,

$V(S_H \geq S_{QM}) = 1.00$, $V(S_H \geq S_{QS}) = 1.00$, $V(S_{QM} \geq S_H) = 0.84$

$V(S_{QM} \geq S_{QS}) = 1.00$, $V(S_{QS} \geq S_H) = 0.47$, and $V(S_{QS} \geq S_M) = 0.61$ are obtained.

Thus, the weight vector from Table 19 is calculated as $W_G = (0.43, 0.37, 0.20)^T$.

From Table 21,

$S_{HM} = (0.32, 0.50, 0.74)$, $S_{HSP} = (0.17, 0.25, 0.39)$, $S_{HSV} = (0.17, 0.25, 0.39)$

$V(S_{HM} \geq S_{HSP}) = 1.00$, $V(S_{HM} \geq S_{HSV}) = 1.00$, $V(S_{HSP} \geq S_{HM}) = 0.21$

$V(S_{HSP} \geq S_{HSV}) = 1.00$, $V(S_{HSV} \geq S_{HM}) = 0.21$, and $V(S_{HSV} \geq S_{HSP}) = 1.00$

are obtained and the weight vector from Table 20 is calculated as $W_H = (0.70, 0.15, 0.15)^T$.

Table 21. Evaluation of the Sub-Attributes with Respect to Hygiene (H)

	HM	HSP	HSV
HM	(1, 1, 1)	(3/2, 2, 5/2)	(3/2, 2, 5/2)
HSP	(2/5, 1/2, 2/3)	(1, 1, 1)	(2/3, 1, 3/2)
HSV	(2/5, 1/2, 2/3)	(2/3, 1, 3/2)	(1, 1, 1)

The weight vector from Table 22 is calculated as $W_{QM} = (0.19, 0.04, 0.77, 0.00)^T$.

Table 22. Evaluation of the Sub-Attributes with Respect to Quality of Meal (QM)

	VM	CoM	CaM	TM
VM	(1, 1, 1)	(3/2, 2, 5/2)	(2/7, 1/3, 2/5)	(5/2, 3, 7/2)
CoM	(2/5, 1/2, 2/3)	(1, 1, 1)	(2/7, 1/3, 2/5)	(7/2, 4, 9/2)
CaM	(5/2, 3, 7/2)	(5/2, 3, 7/2)	(1, 1, 1)	(5/2, 3, 7/2)
TM	(2/7, 1/3, 2/5)	(2/9, 1/4, 2/7)	(2/7, 1/3, 2/5)	(1, 1, 1)

The weight vector from Table 23 is calculated as $W_{QS} = (0.00, 0.05, 0.00, 0.95)^T$.

Table 23. Evaluation of the Sub-Attributes with Respect to Quality of Service (QS)

	BSP	ST	CP	PS
BSP	(1, 1, 1)	(2/7, 1/3, 2/5)	(7/2, 4, 9/2)	(2/9, 1/4, 2/7)
ST	(5/2, 3, 7/2)	(1, 1, 1)	(5/2, 3, 7/2)	(2/7, 1/3, 2/5)
CP	(2/9, 1/4, 2/7)	(2/7, 1/3, 2/5)	(1, 1, 1)	(2/9, 1/4, 2/7)
PS	(7/2, 4, 9/2)	(5/2, 3, 7/2)	(7/2, 4, 9/2)	(1, 1, 1)

The weight vector from Table 24 is calculated as $W_{HM} = (0.66, 0.00, 0.34)^T$.

Table 24. Evaluation of the Catering Firms with Respect to Hygiene of Meal (HM)

	Firm1	Firm2	Firm3
Firm1	(1, 1, 1)	(5/2, 3, 7/2)	(3/2, 2, 5/2)
Firm2	(2/7, 1/3, 2/5)	(1, 1, 1)	(2/7, 1/3, 2/5)
Firm3	(2/5, 1/2, 2/3)	(5/2, 3, 7/2)	(1, 1, 1)

The weight vector from Table 25 is calculated as $W_{HSP} = (0, 0, 1)^T$.

Table 25. Evaluation of the Catering Firms with Respect to Hygiene of Service Personnel (HSP)

	Firm1	Firm2	Firm3
Firm1	(1, 1, 1)	(2/3, 1, 3/2)	(2/9, 1/4, 2/7)
Firm2	(2/3, 1, 3/2)	(1, 1, 1)	(2/5, 1/2, 2/3)
Firm3	(7/2, 4, 9/2)	(3/2, 2, 5/2)	(1, 1, 1)

The weight vector from Table 26 is calculated as $W_{HSV} = (0, 0, 1)^T$.

Table 26. Evaluation of the Catering Firms with Respect to Hygiene of Service Vehicles (HSV)

	Firm1	Firm2	Firm3
Firm1	(1, 1, 1)	(2/3,1, 3/2)	(2/7, 1/3, 2/5)
Firm2	(2/3, 1, 3/2)	(1, 1, 1)	(2/5, 1/2, 2/3)
Firm3	(5/2, 3, 7/2)	(3/2, 2, 5/2)	(1, 1, 1)

The weight vector from Table 27 is calculated as $W_{VM} = (0, 0, 1)^T$.

Table 27. Evaluation of the Catering Firms with Respect to Variety of Meal (VM)

	Firm1	Firm2	Firm3
Firm1	(1, 1, 1)	(2/7, 1/3, 2/5)	(2/3, 1, 3/2)
Firm2	(5/2, 3, 7/2)	(1, 1, 1)	(1, 1, 1)
Firm3	(2/3, 1, 3/2)	(1, 1, 1)	(1, 1, 1)

The weight vector from Table 28 is calculated as $W_{CoM} = (0.87, 0.00, 0.13)^T$.

Table 28. Evaluation of the Catering Firms with Respect to Complementary Meals in a Day (CoM)

	Firm1	Firm2	Firm3
Firm1	(1, 1, 1)	(5/2, 3, 7/2)	(2/3, 1, 3/2)
Firm2	(2/7, 1/3, 2/5)	(1, 1, 1)	(1, 1, 1)
Firm3	(2/3, 1, 3/2)	(1, 1, 1)	(1, 1, 1)

Using the similar calculations, the weight vectors of the catering firms with respect to

Calorie of meal (CaM) is obtained as $W_{CaM} = (0.00, 0.31, 0.69)^T$.

Taste of meal (TM) is obtained as $W_{TM} = (0.27, 0.18, 0.55)^T$.

Behavior of service personnel (BSP) is obtained as $W_{BSP} = (1, 0, 0)^T$.

Service time (ST) is obtained as $W_{ST} = (0.05, 0.64, 0.31)^T$.

Communication on phone (CP) is obtained as $W_{CP} = (0.72, 0.00, 0.28)^T$.

Problem solving ability (PS) is obtained as $W_{PS} = (0, 0, 1)^T$.

Table 29. Obtained Results

Sub-attributes of hygiene					
	HM	HSP	HSV		Alternative priority weight
Weight	0.70	0.15	0.15		
Alternative					
Firm1	0.66	0	0		0.46
Firm2	0	0	0		0.00
Firm3	0.34	1	1		0.54
Sub-attributes of quality of meal					
	VM	CoM	CaM	TM	
Weight	0.19	0.04	0.77	0.00	
Alternative					
Firm1	0	0.87	0	0.27	0.03
Firm2	0	0	0.31	0.18	0.24
Firm3	1	0.13	0.69	0.55	0.73
Sub-attributes of quality of service					
	BSP	ST	CP	PS	
Weight	0.00	0.05	0.00	0.95	
Alternative					
Firm1	1	0.05	0.72	0	0.003
Firm2	0	0.64	0	0	0.032
Firm3	0	0.31	0.28	1	0.965
Main attributes of the goal					
	H	QM	QS		
Weight	0.43	0.37	0.20		
Alternative					
Firm1	0.46	0.03	0.003		0.21
Firm2	0	0.24	0.032		0.10
Firm3	0.54	0.73	0.965		0.69

The combination of priority weights for sub-attributes, attributes, and alternatives to determine priority weights for the best catering firm are shown in Table 29. With respect to the results, firm3 is the catering firm selected.

2.8 Cheng's (1996) Entropy-Based Fuzzy AHP

The Shannon entropy, H, which is applicable only to probability measures, assumes the following form in evidence theory (Klir and Yan, 1995):

$$H(m) = -\sum_{j=1}^{n} m(\{x\}) \log_2 m(\{x\}). \tag{28}$$

This function, which forms the basis of classic information theory, measures the average uncertainty associated with the prediction of outcomes in a random experiment. Its range is

$$[0, \log_2 |X|].$$

$$\text{Clearly}, H(m) = 0.$$

when $m(\{x\}) = 1$ for some $x \in X$; $H(m) = \log_2 |X|$ when m defines the uniform probabilities distribution on $X\left(\text{i.e., } \left(m\{x\}\right) = 1/|X|\ ,\ \forall \text{x} \in \text{X}\right)$.

The principle of maximum uncertainty is well developed and broadly utilized within classic information theory, where it is called the principle of maximum entropy.

Cheng's [1996] evaluation model can be described as given below:

Step 1. Construct a hierarchy structure for any problem.
Step 2. Build membership function of judgment criteria.
Step 3. Compute the performance score.
Step 4. Utilize fuzzy AHP method and entropy concepts to calculate aggregate weights.

The computational procedure of this decision-making methodology is summarized as follows.

To compare the performance scores, we can use symmetric triangular fuzzy numbers $\tilde{1}, \tilde{3}, \tilde{5}, \tilde{7}, \tilde{9}$ to indicate the relative strength of the elements in the hierarchy matrix.

To assemble the total fuzzy judgement matrix $\widetilde{A}$, we can multiply the fuzzy subjective weight vector $\widetilde{W}$ with the corresponding column of fuzzy judgement matrix $\widetilde{X}$. Thus, we get

$$\widetilde{A} = \begin{bmatrix} \widetilde{w}_1 \otimes \widetilde{x}_{11} & \widetilde{w}_2 \otimes \widetilde{x}_{12} & \cdots & \widetilde{w}_n \otimes \widetilde{x}_{1n} \\ \widetilde{w}_1 \otimes \widetilde{x}_{21} & \widetilde{w}_2 \otimes \widetilde{x}_{22} & \cdots & \widetilde{w}_n \otimes \widetilde{x}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \widetilde{w}_1 \otimes \widetilde{x}_{n1} & \widetilde{w}_2 \otimes \widetilde{x}_{n2} & \cdots & \widetilde{w}_n \otimes \widetilde{x}_{nn} \end{bmatrix}. \tag{29}$$

Now fuzzy number multiplications and additions using the interval arithmetic and α cuts are made, and Eq. (30) is obtained.

$$\widetilde{A}_\alpha = \begin{bmatrix} \left[a_{11l}^\alpha, a_{11u}^\alpha\right] & \cdots & \left[a_{1nl}^\alpha, a_{1nu}^\alpha\right] \\ \vdots & \ddots & \vdots \\ \left[a_{n1l}^\alpha, a_{n1u}^\alpha\right] & \cdots & \left[a_{nn1}^\alpha, a_{nnu}^\alpha\right] \end{bmatrix} \tag{30}$$

where $a_{ijl}^\alpha = w_{il}^\alpha x_{ijl}^\alpha$, $a_{iju}^\alpha = w_{iu}^\alpha x_{iju}^\alpha$, for $0 < \alpha \leq 1$ and all i, j.

Now the degree of satisfaction of the judgment $\hat{A}$ will be estimated. When α is fixed, we will set the index of optimism λ by the degree of the optimism of a decision maker. A larger λ indicates a higher degree of optimism. The index of optimism is a linear convex combination it is explained by

$$\hat{a}_{ij}^\alpha = (1-\lambda)\, a_{ijl}^\alpha + \lambda a_{iju}^\alpha, \quad \forall \lambda \in [0,\ 1]. \tag{31}$$

Thus we have

$$\hat{A} = \begin{bmatrix} \hat{a}_{11}^\alpha & \hat{a}_{12}^\alpha & \cdots & \hat{a}_{1n}^\alpha \\ \hat{a}_{21}^\alpha & \hat{a}_{22}^\alpha & \cdots & \hat{a}_{2n}^\alpha \\ \vdots & \vdots & \ddots & \vdots \\ \hat{a}_{n1}^\alpha & \hat{a}_{n2}^\alpha & \cdots & \hat{a}_{nn}^\alpha \end{bmatrix} \tag{32}$$

where $\hat{A}$ is a precise judgment matrix.

The entropy must be first calculated by using the relative frequency of Eq. (33) and the entropy formula of Eq. (34), i.e.,

$$\begin{bmatrix} \frac{a_{11}}{s_1} & \frac{a_{12}}{s_1} & \cdots & \frac{a_{1n}}{s_1} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{a_{n1}}{s_n} & \frac{a_{n2}}{s_n} & \cdots & \frac{a_{nn}}{s_n} \end{bmatrix} = \begin{bmatrix} f_{11} & f_{12} & \cdots & f_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ f_{n1} & f_{n2} & \cdots & f_{nn} \end{bmatrix} \tag{33}$$

where

$$s_k = \sum_{j=1}^{n} a_{kj} .$$

We can use this equation to caloculate the entropy, i.e.,

$$\begin{aligned} H_1 &= -\sum_{j=1}^{n} (f_{1j}) \log_2 (f_{1j}) \\ H_2 &= -\sum_{j=1}^{n} (f_{2j}) \log_2 (f_{2j}) \\ &\vdots \\ H_n &= -\sum_{j=1}^{n} (f_{nj}) \log_2 (f_{nj}) \end{aligned} \tag{34}$$

where H_i is ith entropy value.

The entropy weights can be determined by using Eq. (35).

$$H_i = \frac{H_i}{\sum_{j=1}^{n} H_j}, \quad i = 1, 2, ..., n \tag{35}$$

2.9 A Numerical Example

A company wants to choose one supplier among four suppliers. They determine five attributes: *capacity*, *quality*, *cost*, *distance*, and *delivery time*. By the help of the experts, they determine all the suppliers, and they are given in Table 30. Also for the company they give fuzzy weights of the criteria and they are given in Table 31. This work is done for choosing the best supplier for the company.

Table 30. Fuzzy Judgment Matrix

	X1	X2	X3	X4	X5
A1	(1, 3, 5)	(5, 7, 9)	(3, 5, 7)	(1, 3, 5)	(1, 1, 3)
A2	(7, 9, 9)	(1, 1, 3)	(1, 3, 5)	(1, 3, 5)	(3, 5, 7)
A3	(1, 1, 3)	(1, 3, 5)	(1, 3, 5)	(3, 5, 7)	(1, 3, 5)
A4	(3, 5, 7)	(5, 7, 9)	(1, 1, 3)	(1, 1, 3)	(1, 3, 5)

Table 31. Fuzzy Subjective Weight Vector

	X1	X2	X3	X4	X5
W	(1, 3, 5)	(7, 9, 9)	(5, 7, 9)	(1, 3, 5)	(1, 1, 3)

The following phases are taken to solve the problem:

To assemble the total fuzzy judgment matrix $\widetilde{A}$, we can multiply the fuzzy subjective weight vector $\widetilde{W}$ by the corresponding column.

$\widetilde{A} =$

	X1	X2	X3	X4	X5
A1	(1, 3, 5) ⊗ (1, 3, 5)	(7, 9, 9) ⊗ (5, 7, 9)	(5, 7, 9) ⊗ (3, 5, 7)	(1, 3, 5) ⊗ (1, 3, 5)	(1, 1, 3) ⊗ (1, 1, 3)
A2	(1, 3, 5) ⊗ (7, 9, 9)	(7, 9, 9) ⊗ (1, 1, 3)	(5, 7, 9) ⊗ (1, 3, 5)	(1, 3, 5) ⊗ (1, 3, 5)	(1, 1, 3) ⊗ (3, 5, 7)
A3	(1, 3, 5) ⊗ (1, 1, 3)	(7, 9, 9) ⊗ (1, 3, 5)	(5, 7, 9) ⊗ (1, 3, 5)	(1, 3, 5) ⊗ (3, 5, 7)	(1, 1, 3) ⊗ (1, 3, 5)
A4	(1, 3, 5) ⊗ (3, 5, 7)	(7, 9, 9) ⊗ (5, 7, 9)	(5, 7, 9) ⊗ (1, 1, 3)	(1, 3, 5) ⊗ (1, 1, 3)	(1, 1, 3) ⊗ (1, 3, 5)

$$\forall_\alpha = [0,1], \quad \widetilde{A}_\alpha = \left[a_L^\alpha, \mathrm{a}_\mathrm{R}^\alpha\right] = \left[(a_2 - a_1)\alpha + a_1,\ -(a_3 - a_2)\alpha + a_3\right]$$

Set α = 0.8 and λ = 0.5 for a moderate decision maker.

$\hat{a}_{11}$ = [(3 − 1) × 0.8 + 1, − (5 − 3) × 0.8 + 5] ⊗ [(3 − 1) × 0.8 + 1, − (5 − 3) × 0.8 + 5]

$\hat{a}_{11}$ = [6.76, 11.56]

All the results are given below.

$\widetilde{A}_{\alpha=0.8} =$

	X1	X2	X3	X4	X5
A1	[6.76, 11.56]	[56.76, 66.6]	[30.36, 39.96]	[6.76, 11.56]	[1, 1.96]
A2	[22.36, 30.6]	[8.6, 12.6]	[17.16, 25.16]	[6.76, 11.56]	[4.6, 7.56]
A3	[2.6, 4.76]	[22.36, 30.6]	[17.16, 25.16]	[11.96, 18.36]	[2.6, 4.76]
A4	[11.96, 18.36]	[56.76, 66.6]	[6.6, 10.36]	[2.6, 4.76]	[2.6, 4.76]

where $\alpha = 0.8$.

We compute to $\hat{a}_{11}^{\alpha}$ by using Eq. (31) as

$$\hat{a}_{11}^{\alpha} = (1 - 0.5) \times 6.76 + 0.5 \times 11.56 = 9.16$$

All the results are given below.

$\hat{A} =$

	X1	X2	X3	X4	X5
A1	9.16	61.68	35.16	9.16	1.48
A2	26.48	10.6	21.16	9.16	6.08
A3	3.68	26.48	21.16	15.16	3.68
A4	15.16	61.68	8.48	3.68	3.68

where $\lambda = 0.5$.

We calculate relative frequencies by Eq. (33).

$f =$

	X1	X2	X3	X4	X5
A1	0.0785	0.5288	0.3014	0.0785	0.0126
A2	0.3603	0.1442	0.2879	0.1246	0.0827
A3	0.0524	0.3774	0.3015	0.2160	0.0524
A4	0.1635	0.6655	0.0914	0.0397	0.0397

Then, we compute entropy values by using the relative frequencies and the entropy formula Eq. (34). The resultant aggregate weights can be determined by normalizing entropy values.

	Entropy Value	Entropy Weight
A1	H1 = 1.6635	0.2289
A2	H2 = 2.1225	0.2921
A3	H3 = 1.9754	0.2719
A4	H4 = 1.5053	0.2069

From the last table, supplier A_2 is the best choice when $\alpha = 0.8$ and $\lambda = 0.5$.

3. CONCLUSION

Decisions are made today in increasingly complex environments. The fuzzy AHP provides a systematic method for comparison and weighting of the multiple criteria and alternatives to decision makers in the case of incomplete information. Many alternative fuzzy AHP methods exist in the literature, whereas only a crisp does. The problem that one is superior to any other has not yet been solved. Besides it is possible to meet new alternative fuzzy AHP methods in the near future. These new methods should try to follow the fundamentals of the crisp AHP. Otherwise, these methods could not be called AHP-based multi-criteria decision-making methods.

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A SWOT-AHP APPLICATION USING FUZZY CONCEPT: E-GOVERNMENT IN TURKEY

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Abstract: E-government refers to the delivery of information and services online via the Internet. Many governmental units across the world have embraced the digital revolution and placed a wide range of materials on the web, from publications to databases. The purpose of this study is to evaluate and to determine the alternative strategies for e-government applications in Turkey. We use the strengths, weaknesses, opportunities, and threats (SWOT) approach in combination with the crisp and fuzzy analytic hierarchy process (AHP) to achieve this task. The strategies have been prioritized by using both methods comparatively and sensitivity analyses of the obtained results have been presented.

Key words: Outranking, fuzzy outranking relation, pair-wise comparison, e-government, SWOT, analytic hierarchy process, strategic planning, sensitivity analysis

1. INTRODUCTION

Digital technologies serve as a basic source of transformation in economies, communities, and government functions all over the world. The occurrence of technological change of the late 1990s, the result of which was the enabling of the delivery of services over the internet, caused major and rapid transformation of how governments function. Development of e-commerce and the evolution projected for the near future has encouraged consumers to demand more and more customized, rapid, and at-home services. The rapid development of modern information and communication technologies is having far-reaching effects on all aspects of modern life, including government.

Academics have suggested various definitions for e-government. According to Kaylor et al. (2001), e-government is taken to be the ability for citizens to communicate and/or interact with the city via the Internet in any way more sophisticated than a simple e-mail letter to the generic city (or webmaster) or e-mail address provided at the site. The United Nations and the American Society for Public Administration (2002) defined e-government as "utilizing the internet and the World-Wide-Web for delivering government information and services to citizens." More recently, e-government is defined by the OECD (2003) as "the use of Information and Communications Technologies (ICT), and particularly the Internet, as a tool to achieve better government."

Many works on e-government have been published. Most of these works are on strategy evaluation, future development programs, and scenario planning. Layne and Lee (2001) described different stages of e-government development. The stages of development outline the structural transformations of government as they progress toward electronically enabled government. And they also described how the Internet-based government emerged with traditional models amalgamated with traditional public administration, implying fundamental changes in the form of government. They developed a four-stage growth model with themselves providing observation. Chen and Gant (2001) examined the potential of application service providers to transform electronic government services at the local level. Gupta and Jana (2003) suggested a flexible framework to choose an appropriate strategy to measure the tangible and intangible benefits of e-government. Reddick (2004) explored the current stages of development and prospect for future development in e-government growth in the U.S. cities. Akman et al. (2005) reviewed and discussed e-government issues in general, its global perspective, and then reported the findings of a survey concerning impact of gender and education among the e-government users in Turkey. Gil-Garcia and Pardo (2005) examined the extent to which information systems (IS) research informs the development of practitioner tools for government information technology (IT) decision makers.

SWOT, the acronym standing for strengths, weaknesses, opportunities and threats analysis, is a commonly used tool for analyzing internal and external environments to attain a systematic approach and support for a decision situation. The internal and external factors most important to the enterprise's future are referred to as strategic factors, and they are summarized within the SWOT analysis. The final goal of a strategic planning process, of which SWOT is an early stage, is to develop and adopt a strategy resulting in a good fit between internal and external factors. SWOT can also be used when strategy alternative emerges suddenly and when the decision context relevant to it has to be analyzed. This chapter proposes a multi-attribute decision-making-based SWOT analysis for the evaluation of alternative e-government strategies for Turkey. After a wide literature review, it is found that the application of this methodology to e-government area is described for the first time in this chapter.

In this chapter, we applied a SWOT analysis using the crisp and the fuzzy approaches of a multi-attribute evaluation method that is called the analytic hierarchy process (AHP) to the e-government process of Turkey. E-government strategy selection with SWOT analysis is a complex problem in which many qualitative aspects must be considered. These kinds of aspects make the evaluation process hard and vague. The judgments from experts are always vague and linguistic rather than exact values. Thus, it is suitable and flexible to express the judgments of experts in fuzzy quantities instead of in crisp quantities. Additionally, the hierarchical structure is a good approach to describe these kinds of complicated evaluation problems. Fuzzy AHP has the capability of taking these situations into account with a hierarchical structure. To be able to compare with the crisp case, we also implemented the crisp AHP. We first determined the factors in the SWOT groups and alternatives strategies for e-government application in Turkey. Then we computed the importance weights of these factors and the scores of the strategies. The aim of this study is to determine the priorities of the e-government strategies for the case of Turkey.

The remainder of this chapter is organized as follows. Section 2 presents Turkey's position among the other countries and e-government projects and services in Turkey. Section 3 introduces some terminology from SWOT analysis and the analytic hierarchy process (crisp and fuzzy cases). The method for utilizing AHP in SWOT analysis is also defined in the third section. SWOT analysis for e-government in Turkey is presented in Section 4. Sections 5 and 6 define possible e-government strategies and the evaluation of e-government strategies in Turkey. The last section summarizes the findings and makes suggestions for further research.

2. E-GOVERNMENT IN TURKEY

2.1 Turkey's Position Among the Other Countries

The e-government agenda is being pursued throughout the world to one degree or another, but it has added significance in Central Europe. The region is just beginning to emerge from a period of far-reaching political and economic transformation after the collapse of repressive communist systems. For these countries, e-government is more than simply a new channel of delivering services; it offers an opportunity to achieve a quantum leap in transparency and efficiency of administration, which the region's leaders have promised their citizens since the early 1990s. In order to gauge their capacity to implement such change as well as their progress to date, the Economist Intelligence Unit (EIU), sponsored by Oracle, conducted a wide-ranging analysis of the e-government experience in the Central Europe region. EUI considered seven criteria with different weightings: connectivity and technology infrastructure (CTI) (20% weight), business and legal environment (BLE) (10% weight), e-democracy (E-D) (15%), education and skills (ES) (10%), online public services for citizens (OPSC) (15%), online public service for businesses (OPSB) (15%), and government policy and vision (GPV) (15%). Table 1 expresses the results of this analysis in comparative fashion. Scores are on a scale of 1 to 10 (*Source*: Economist Intelligence Unit). The rankings cover the ten new and candidate EU members from Central Europe, as well as another prospective member, Turkey.

Table 1. Economist Intelligence Unit Central Europe e-government rankings

	Overall score	CTI	BLE	ES	GPV	E-D	OPSC	OPSB
Category weight		0.20	0.10	0.10	0.15	0.15	0.15	0.15
Estonia	5.87	3.37	6.80	7.67	6.50	4.60	6.38	7.52
Czech Rep.	5.67	3.98	6.95	7.33	6.10	3.60	5.68	7.57
Slovenia	5.33	3.68	6.60	7.33	5.00	2.90	6.73	6.68
Poland	4.74	2.43	6.60	6.67	5.30	2.90	5.98	5.33
Hungary	4.69	3.15	6.66	7.00	5.50	3.30	5.00	4.19
Turkey	4.64	2.67	4.23	5.67	4.90	4.20	5.70	6.00
Lithuania	4.62	2.21	6.36	6.33	4.70	2.60	5.00	7.08
Latvia	4.58	2.34	6.32	6.67	5.00	2.60	4.79	6.35
Slovakia	4.44	2.80	6.28	6.67	3.80	2.90	4.46	6.08
Romania	3.99	1.43	5.42	5.33	4.70	2.60	4.08	6.16
Bulgaria	3.71	1.92	5.50	5.67	3.10	2.60	3.95	5.08

2.2 Process of Turkey's Development in e-Government

Since the beginning of the 1990s, there has been an increase in the effort by most countries to transform into an information society. Essentially, economic and social necessities bring about these efforts. The Turkish Government initiated the Urgent Action Plan in December 2002 to remedy long-lasting economic problems and to improve the social well-being of the country. One basic component of this plan is the "e-Transformation Turkey Project," which aims to turn Turkey into an information society. Some objectives of this project are to facilitate the participation of citizens to the decision-making process; to enhance transparency and accountability for the public management; to promote ICT diffusion; and to coordinate e-government investments by means of information and communication technologies. The Minister of State and Deputy Prime Minister has the high level responsibility of the project, and the project is coordinated by the State Planning Organization (SPO).

In order to realize the objectives of this project and to ensure the success of the project, a new coordination unit, the *Information Society Department*, within SPO was established. This department is responsible for the overall coordination of the project. Before this project was launched, lack of efficient coordination between institutions made the progress slow and ineffective. For the first time in Turkey, a dedicated department, which is believed to be a crucial element for success, has been named as the coordinator of information society activities. To increase the participation and the level of success, an Advisory Committee with 41 members has been established. This consulting body consists of the representatives of public institutions, nonprofit organizations, and universities (Akman et al., 2005)

In line with the government's schedule, the initial focal point in this project was the *Short Term Action Plan (STAP)*, which covered 2003−2004, for implementing specific tasks. The first action of STAP was the determination of an "Information Society Strategy," which encompassed every part of society and maximized national benefits and value added. As in the preparation phase, the implementation of STAP and all other related activities was coordinated by the SPO-Information Society Department and was open to every contribution in order to successfully achieve the ultimate goal: to transform Turkey into an information society.

The e-Transformation Turkey Executive Board was also established with the same circular that validates STAP. The Board is composed of the Minister of State and Deputy Prime Minister (e-minister), Minister of

Industry and Trade, Minister of Transport, Undersecretary of SPO, and Chief Advisor to the Prime Minister. The Board was given the responsibility of supervision of the e-Transformation Turkey Project.

2.3 E-Government Projects and Services In Turkey

Akman et al. (2002) classify and report a list of current e-government projects. A classification based on the project characteristics is given as follows: Projects related to national IS and services; projects related to education, culture, youth, and sports; projects related to health, family, labor, and social affairs; projects related to finance and economics; projects related to interior affairs; projects related to justice affairs; projects related to agriculture, forestry, village, and environmental affairs; projects related to industry, technology, energy, and natural resources; projects related to communications, public work, tourism, and development and housing; and projects related to foreign affairs. From the details of the report, one can observe that most of the projects are for processing information and hence devoted to services. However, a considerable number of these systems is being developed for government-to-government (G2G) communications. Although these systems do not adopt the government-to-citizens (G2C) approach entirely, the above classification provides evidence that these will constitute a sufficient base for G2C communication in the future in almost all areas of public affairs. Currently available e-government services in Turkey are classified into three groups. G2G services are the ones enjoyed among public organizations electronically. G2C services are given by government organizations to citizens electronically. Government-to-business (G2B) services are given by government organizations to private sector.

3. SWOT-AHP ANALYSIS FOR E-GOVERNMENT

In the following discussion, the fundamentals of SWOT analysis and AHP are given. Later, these techniques are combined to prioritize the e-government strategies.

3.1 SWOT Analysis

A scan of the internal and external environment is an important part of the strategic planning process. Environmental factors internal to the organization usually can be classified as strengths (S) or weaknesses (W), and those external to the organization can be classified as opportunities (O) or threats (T). Such an analysis of the strategic environment is called to as a SWOT analysis. The SWOT approach involves systematic thinking and comprehensive diagnosis of factors relating to a new product, technology, management, or planning (Weihrich, 1982). Figure 1 shows how SWOT analysis fits into an environment scan.

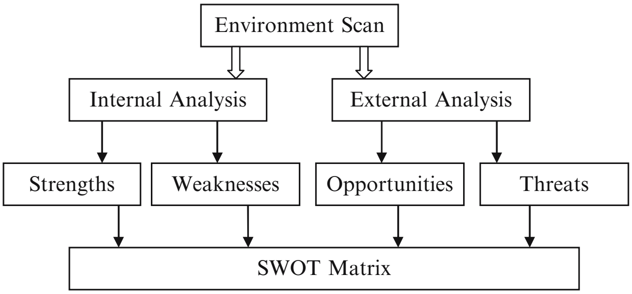

Figure 1. SWOT analysis framework

3.2 A Multi-Attribute Evaluation Method: AHP

The analytic hierarchy process has been used in many different fields as a multi-attribute decision analysis tool with multiple alternatives and criteria. An extensive literature review on AHP can be found in Vaidya and Kumar's (2006) study. AHP uses "pair-wise comparisons" and matrix algebra to weigh criteria. The decision is made by using the derived weights of the evaluative criteria (Saaty, 1980).

After the hierarchy of the problem is constructed, the matrices of pair-wise comparisons are obtained. In this matrix, the element $a_{ij} = 1/a_{ij}$, and thus, when $i = j$, $a_{ij} = 1$. The value of w_i may vary from 1 to 9, and 1/1 indicates equal importance, whereas 9/1 indicates extreme or absolute importance. The scale is shown in the Table 2.

Table 2. Evaluation Scale

Num Value	Verbal Scale
1	Equal importance
3	Moderate importance
5	Strong importance
7	Very strong importance
9	Extreme or absolute importance
2, 4, 6, 8	Intermediate values

$$A = (a_{ij}) = \begin{bmatrix} 1 & w_1/w_2 & \cdots & w_1/w_n \\ w_2/w_1 & 1 & \cdots & w_2/w_n \\ \vdots & \vdots & \cdots & \vdots \\ w_n/w_1 & w_n/w_2 & \cdots & 1 \end{bmatrix} \tag{1}$$

In the comparisons, some inconsistencies can be expected and accepted. When A contains inconsistencies, the estimated priorities can be obtained by using the A matrix as the input using the eigenvalue technique.

$$(A - \lambda_{max} I)q = 0 \tag{2}$$

where $\lambda_{\max}$ is the largest eigenfactor of matrix A of size n, q is its correct eigenfactor and I is the identity matrix of size n. The correct eigenfactor, q, constitutes the estimation of relative priorities. Each eigenfactor is scaled to sum up to one to obtain the priorities. Saaty (1977) demonstrated that $\lambda_{\max} = n$ is a necessary and sufficient condition for consistency. Inconsistency may occur when $\lambda_{\max}$ deviates from n due to inconsistent responses in pair-wise comparisons. Therefore, the matrix A should be tested for consistency using index, *CI*, which has been constructed.

$$\mathrm{CI} = (\lambda_{max} - n)/(n-1) \tag{3}$$

CI estimates the level of consistency with respect to a comparison matrix. Then, because *CI* is dependent on n, a consistency ratio *CR* is calculated, which is dependent of n as shown below.

$$\mathrm{CR} = \mathrm{CI} / \mathrm{RI} \tag{4}$$

where CI is the consistency index, *RI* is random index (RI) generated for a random matrix of order n, and CR is the consistency ratio (Saaty, 1993). The general rule is that $\mathrm{CR} \leq 0.1$ should be maintained for the matrix to be consistent. Otherwise, all or some comparisons must be repeated in order to resolve the inconsistencies of the pair-wise comparisons.

3.3 The Fuzzy AHP

To deal with vagueness of human thought, Zadeh (1965) first introduced the fuzzy set theory, which was oriented to the rationality of uncertainty due to imprecision or vagueness. A major contribution of fuzzy set theory is its capability of representing vague data. The theory also allows mathematical operators and programming to apply to the fuzzy domain. A fuzzy set is a class of objects with a continuum of grades of membership. Such a set is characterized by a membership (characteristic) function, which assigns to each object a grade of membership ranging between zero and one.

A tilde "~" will be placed above a symbol if the symbol represents a fuzzy set. A triangular fuzzy number (TFN), $\tilde{M}$ is shown in Figure 2. A TFN is denoted simply as $(l/m, m/u)$ or (l, m, u). The parameters *l*, *m*, and *u* respectively, denote the smallest possible value, the most promising value, and the largest possible value that describe a fuzzy event.

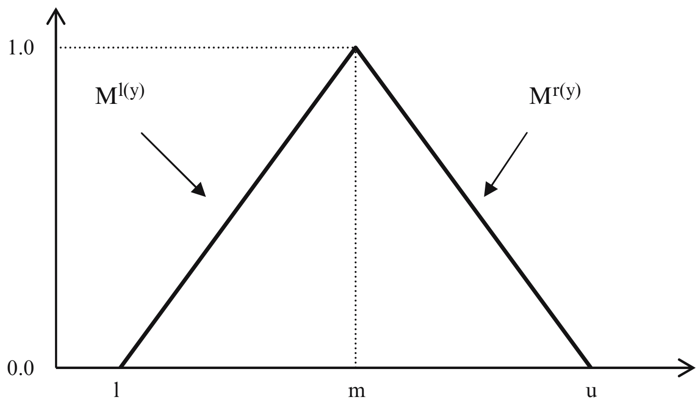

Figure 2. A Triangular Fuzzy Number, $\tilde{M}$

Each TFN has linear representations on its left and right side such that its membership function can be defined as

$$\mu\left(x/\tilde{M}\right) = \begin{cases} 0, & x < l \\ (x-l)/(m-l), & l < x < m \\ (u-x)/(u-m), & x > l \end{cases} \tag{5}$$

A fuzzy number can always be given by its corresponding left and right representation of each degree of membership:

$$\tilde{M} = (M^{l(y)}, M^{r(y)} = \left(l + (m-l)y, u + (m-u)y\right) \qquad y \in [0, 1] \tag{6}$$

where $l(y)$ and $r(y)$ denote the left side representation and the right side representation of a fuzzy number, respectively. Many ranking methods for fuzzy numbers have been developed in the literature. These methods may give different ranking results, and most methods are tedious in graphic manipulation requiring complex mathematical calculation. The algebraic operations with fuzzy numbers can be found in Zimmermann (1994).

A basic literature review on fuzzy AHP can be found in Kahraman et al.'s (2004) study. In this chapter, we prefer Chang's (1992, 1996) extent analysis method since the steps of this approach are relatively easier than the other fuzzy AHP approaches and similar to the conventional AHP. This method can be found with its details in Chapter 3, Section 3.2.6.

3.4 The Method for Using AHP in SWOT Analysis

The idea in using AHP within a SWOT framework is to systematically evaluate SWOT factors and commensurate their intensities. If it is used in combination with the analytic hierarcy process, the SWOT approach can provide a quantitative measure of importance of each factor on decision making (Saaty and Vargas, 2001). The method introduced proceeds as follows (Kurttila et al., 2000):

Step 1. SWOT analysis is carried out.

The relevant factors of the external and internal environments are identified and included in SWOT analysis. When standard AHP is applied, it is recommended that the number of factors within a SWOT group should not exceed 10 because the number of pair-wise comparisons needed in the analysis increases rapidly (Saaty, 1980). Thus, the result of the comparisons is quantitative values expressing the priorities of the factors included in SWOT analysis.

Step 2. Pair-wise comparisons between SWOT factors are carried out within every SWOT group.

When making the comparisons, the questions at stake are as follows: (1) which of the two factors compared is a greater strength (opportunity, weakness, or threat); and (2) how much greater. With these comparisons as the input, the relative local priorities of the factors are computed using the eigenvalue method. These priorities reflect the decision maker's perception of the relative importance of the factors.

Step 3. Pair-wise comparisons are made among the four SWOT groups.

The factor with the highest local priority is chosen from each group to present the group. These four factors are then compared as in Step 2. These are the scaling factors of the four SWOT groups, and they are used to calculate the global priorities of the independent factors within them. This is done by multiplying the factors' local priorities (defined in Step 2) by the value of the corresponding scaling factor of the SWOT group. The global priorities of all factors sum up to one.

Step 4. The results are used in the strategy formulation and evaluation process.

The contribution to the strategic planning process comes in the form of numerical values for the factors. New goals may be set, strategies may be defined, and such implementations may be planned as take into close consideration the foremost factors.

4. SWOT ANALYSIS FOR E-GOVERNMENT IN TURKEY

In the following discussion, we determine the subfactors of the strengths, weaknesses, opportunities, and threats for e-government in Turkey. These subfactors are used in the prioritization of the e-government strategies.

4.1 Strengths

The three main strengths are determined as follows.

4.1.1 Formation of Supervisory and Executive Committees

The Prime Minister of Turkey made a declaration that was published in the October 4, 2003, issue of the Official Gazette for the realization of the STAP covering the years 2003 and 2004 for the e-Transformation project. The declaration specified the tasks along with their priorities, the formation of the supervisory and executive committees, and the responsible organizations in charge of implementation. The Supervisory committee is headed by the Deputy Prime Minister and includes members from top−level management of the public and private sectors and the NGO representatives. The members of the Executive Committee are Deputy Prime Minister (Chair), Minister of Industry and Trade, Minister of Transportation, Undersecretary of the SPO, and Head Advisor of the Prime

Minister. The representatives of nongovernmental organizations, Trade Chambers Union, Informatics Association of Turkey, Turkish Informatics Foundation, The Association of Turkish IT Industrialists, the head of the Telecommunications Authority, and the general manager of Turkish Telecom are also members of this Committee. Some of the major achievements realized by the committee have been the settlement of three laws, namely, the e-Signature law, Knowledge Acquisition law, and Privacy law (Akman et al., 2005).

4.1.2 e-Transformation Projects

The Turkish Government has started many e-government projects as indicated in Section 4.2.4. These projects are G2G, G2C, and G2B services. They are the main locomotives of e-government in Turkey.

4.1.3 Support from Top–Level Management of the Public and Private Sectors

Both public and private sectors support the e-government projects. Many ministers from the government and many associations and foundations are the members of the e-government committees.

4.2 Weaknesses

The four main weaknesses are determined as follows.

4.2.1 Lack of Access to Internet Among Certain Sections of The Population

It is a particular problem for public–sector organizations, as they cannot choose their customers. Indeed many public services are provided specifically for vulnerable or low-income groups who are the least likely to have access to the technology. The main consequence is that public–sector organizations will have to continue to provide services through multiple channels at least in the short term to prevent excluding those who do not have access to the Internet (Akman et al., 2005).

4.2.2 Lack of Finance for Capital Investment in New Technologies

The reason is that IT was often not viewed as a priority when competing for scare resources against other claims for capital investment, for example, new schools, roads, and so on.

4.2.3 Need To Change Individual Attitudes and Organizational Cultures

It is part of an organizational change issue. Another problem is with security and authentication that prevented the development of electronic transaction services. It is a specific problem with public−sector organizations as the public generally saw them as being in a position of trust.

4.2.4 Poor Economic Power of Citizens and Businesses

Although gross national product per person was US$3,412 in 2003, it was US$4,172 in 2004. This increase means US$348 per month for one person. It is clear that citizens having this level of economic power cannot buy a computer and other software and hardware.

4.3 Opportunities

Nine subfactors of opportunities for e-government in Turkey are determined. The first three subfactors are selected by the experts for the SWOT analysis since these are evaluated as the most important issues of e-government in Turkey.

4.3.1 A Candidate Country from the European Union's Information Society Perspective

Turkey has been a candidate country according to the European Union (EU) for a long time. Being a member of the EU will force Turkey to implement e-government conditions. So we accept it as an opportunity for Turkey.

4.3.2 Efficiency

As with many information technology-related projects, one of the anticipated benefits is improved efficiency. In e-government projects, this efficiency can take many forms. Some projects seek to reduce errors and improve consistency of outcomes by automating standardized tasks. A related efficiency goal of many e-government initiatives is to reduce costs and layers of organizational processes by re-engineering and streamlining operating procedures.

4.3.3 New and Improved Services

Another opportunity promoted by e-government supporters is the potential to improve the quality, range, and accessibility of services. Some observers suggest that, in addition to enhanced efficiency, the quality of services may be improved through quicker transactions, improved accountability, and better processes. The evolution of e-government also creates the potential for new services.

4.3.4 Increased Citizen Participation

A third benefit anticipated by some e-government advocates is increased citizen participation in government. One way this could occur is by connecting people who live in remote areas of the country so that they can send and receive information more easily. A second way suggested by some observers is through increased participation in government by younger adults.

4.3.5 Improved National Information Infrastructure

A fourth possible benefit of the drive to implement e-government initiatives is the improvement of the national information infrastructure.

4.3.6 Potential Challenges to e-Government

On the other hand, despite the potential opportunities for the implementation of e-government initiatives, several challenges that could prevent the realization of these anticipated benefits. Some of the challenges, such as disparities in computer access (digital divide—the lack of equal access to computers, whether due to a lack of financial resources or necessary skills), are preexisting conditions that are connected to larger issues.

4.3.7 Privacy

Related to computer security, privacy also presents a challenge to the implementation and acceptance of e-government initiatives. Concerns about the use of "cookies," sharing information between agencies (computer matching), and the disclosure or mishandling of private information are frequent subjects of debate. Addressing the issue of privacy in the context of e-government may require both technical and policy responses.

4.3.8 Disparities in Computer Access

Another challenge for e-government are disparities in computer access. This challenge includes two policy issues: the often described "digital divide" and accessibility for people with disabilities. In the case of the digital divide, not all citizens currently have equal access to computers, whether due to a lack of financial resources or necessary skills. Although the placement of Internet-enabled computers in schools and public libraries is helping address this issue, these efforts are still progressing.

4.3.9 Government Information Technology Management and Funding

A multilayered challenge for the development of e-government is government information technology management and funding, which includes issues such as government information technology worker recruitment, retention, and compensation and cooperation between local, state, and federal governments.

4.4 Threats

The four main threats are determined as follows.

A. Decentralized Internet Governance

Various bodies and companies assert control over parts of the Internet and try to exercise it through technical and political means. This threatens the Internet's stability and usability.

B. Copyright Lawsuits

Some people use the Internet to trade copyrighted works, particularly music, video, and software. Copyright owners object to this practice and attempt to discourage the practice through highly public legal action against participants.

C. Inadequate Government IT Security

Poor safeguarding of personal information could damage the uptake of government services online. For example, on rare occasions, personal records have been found at landfill sites, which have caused concern. If people are concerned about government security and about Internet security, they are doubly unlikely to use e-government services.

D. Inadequate Government IT Security

Concerns are sometimes raised about the availability of "dangerous" information, such as bomb-making recipes, on the Internet, and about disinformation or opinion being presented as fact.

4.5 SWOT-AHP Analysis

Evaluations are made by four experts in a group meeting and they are asked to make a collective group decision both on an evaluation score (Table 2) and on a linguistic term describing the comparison of SWOT factors and subfactors.

4.5.1 Crisp SWOT-AHP Analysis

When the analysis has been completed, a SWOT matrix can be generated and used as a basis for goal setting, strategy formulation, and implementation. The subfactors of SWOT analysis are placed in a SWOT matrix as shown in Figure 3.

STRENGTHS	OPPORTUNITIES
* S1: Settlement of three laws, namely, e-Signature law, Knowledge Acquisition law, and Privacy law. * S2: e-Transformation projects. * S3: Supports from top−level managements of public and private sectors and the NGO representative.	* O1: A candidate country from the European Union's information society perspective. * O2: Efficiency by reducing costs and layers of organizational processes by re-engineering. * O3: New and improved services.
WEAKNESSES	**THREATS**
* W1: Lack of access to Internet among certain sections of the population. * W2: Lack of finance for capital investment in new technologies. * W3: Need to change individual attitudes and organizational cultures. * W4: Poor economic power of citizens and businesses.	* T1: Decentralized internet governance. * T2: Inadequate government IT security. * T3: Copyright lawsuits. * T4: Availability of "Dangerous" Information.

Figure 3. SWOT Matrix

In the following discussion, the pair-wise comparison matrix among SWOT groups and an instance of the comparison matrices of the subfactors are given (Tables 4 and 5). The sample pair-wise comparisons for each level of the hierarchy are given in the Appendix.

Table 3. Pair-wise Comparison Matrix of the SWOT Groups

With respect to the goal	Strengths	Weaknesses	Opportunities	Threats
Strengths	1	1/3	1/7	5
Weaknesses	3	1	1/4	7
Opportunities	7	4	1	9
Threats	1/5	1/7	1/9	1

Table 4. Pair-wise Comparison Matrix of the Strengths Criteria

With respect to strengths group	S1	S2	S3
S1	1	1/7	1/3
S2	7	1	5
S3	3	1/5	1

Using the pair-wise comparison matrices given above and the Expert Choice software package, the priorities of the SWOT groups and the subfactors, which are shown in Table 6 have been obtained.

Table 5. Priorities and Consistency Ratios of Comparisons of the Swot Groups and Sub Factors

SWOT group	Priority of the group	SWOT factors	Inconsistency ratio	Priority of the factor within the group	Overall priority of the factor
Strengths	0.109	*S1. Formation of supervisory and executive committees*	0.06	0.081	0.009
		S2. e-Transformation projects		0.731	0.077
		S3. Support from top−level management of the public and private sectors		0.188	0.020
Weaknesses	0.230	*W1. Lack of access to Internet among certain sections of the population*	0.10	0.088	0.022
		W2. Lack of finance for capital investment in new technologies		0.636	0.162
		W3. Need to change individual attitudes and organizational cultures,		0.041	0.010
		W4. Poor economic power of citizens and businesses		0.235	0.060
Opportunities	0.623	*O1. A candidate country for the European Union's information society perspective,*	0.06	0.731	0.438
		O2. Efficiency by reducing costs and layers of organizational processes by re-engineering		0.081	0.049
		O3. New and improved services.		0.188	0.113
Threats	0.038	*T1. Decentralized Internet governance,*	0.05	0.048	0.002
		T2. Inadequate government IT security		0.658	0.027
		T3. Copyright lawsuits		0.083	0.003
		T4. Availability of "Dangerous" Information		0.212	0.009

Figure 4 illustrates the priority weights of the categorized subfactors whose numerical values are given in Table 6.

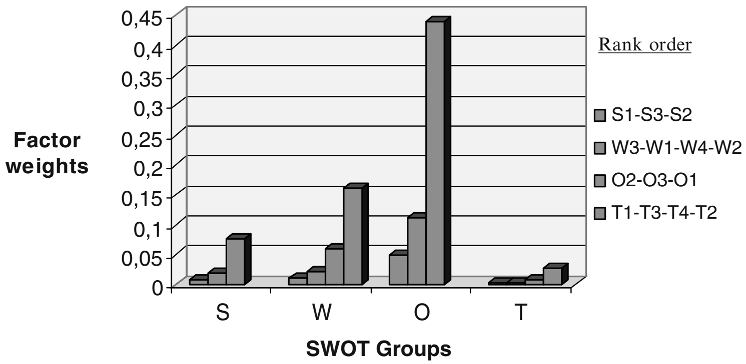

Figure 4. The priority weights of the categorized subfactors with crisp AHP

Table 6. The Fuzzy Pair-wise Comparison Matrix of the SWOT Groups

GOAL	S	W	O	T
S	(1, 1, 1)	(1/2, 2/3, 1)	(2/5, 1/2, 2/3)	(1, 3/2, 2)
W	(1, 3/2, 2)	(1, 1, 1)	(2/5, 1/2, 2/3)	(1, 3/2, 2)
O	(3/2, 2, 5/2)	(3/2, 2, 5/2)	(1, 1, 1)	(3/2, 2, 5/2)
T	(1/2, 2/3, 1)	(1/2, 2/3, 1)	(2/5, 1/2, 2/3)	(1, 1, 1)

Figure 5 illustrates the graphical interpretation of the results of pair-wise comparisons for SWOT groups and factors. The whole situation is easily observed by referring to Figure 5. The lengths of the lines in the different sectors point out that the weaknesses and opportunities predominate and that currently no specific strengths and threats could ruin the new strategy.

Figures 6 and 7 show the results of the sensitivity analysis with respect to the goal.

From Figure 6, we see the overall weights on the right side of the figure, indicating that *S2* (*e-Transformation projects*) is the most important subfactor of all. When we increase the weight of the *strengths* group to make it the largest of all the groups, as illustrated on the *strengths* line, the rank order is *S2-T2-W2-S3-O1-T4-W4-S1-T3-O3-W1-T1-O2-W3*. From Figure 7, we see the overall weights on the right side of the figure, indicating that *W2* (*Lack of finance for capital investment in new technologies*) is the most important subfactor of all. When we increase the weight of the *weaknesses* group to make it the largest of all the groups, as illustrated on *weaknesses* line, the rank order is *W2-O1-W4-O3-W1-S2-O2-W3-T2-S3-T4-S1-T3-T1*.

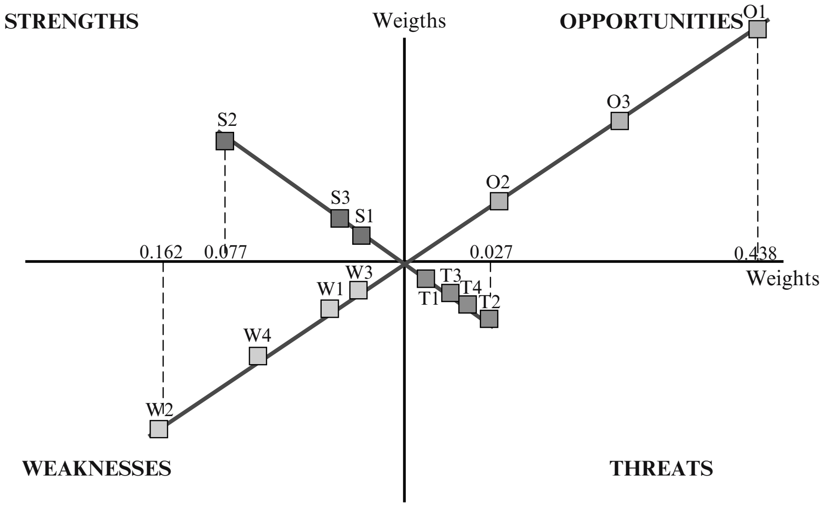

Figure 5. Graphical interpretation of the results of pair-wise comparisons of SWOT groups and factors

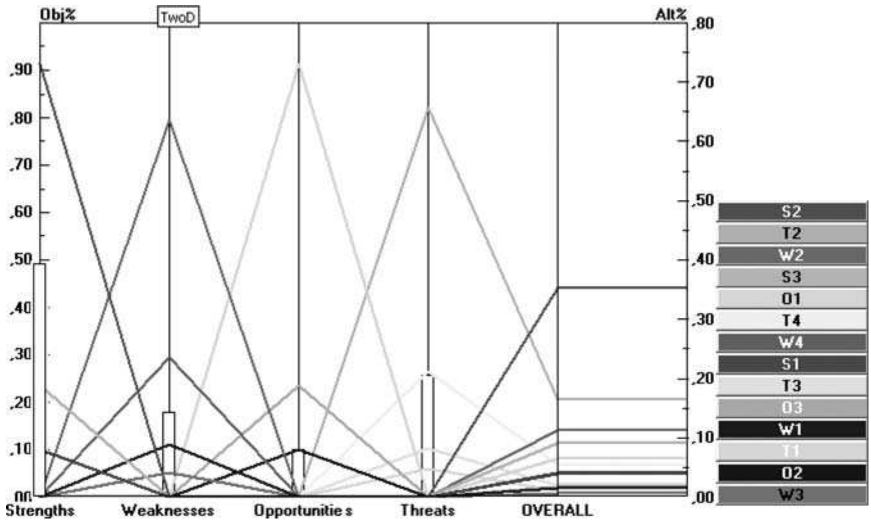

Figure 6. Sensitivity with respect to the goal: SWOT groups

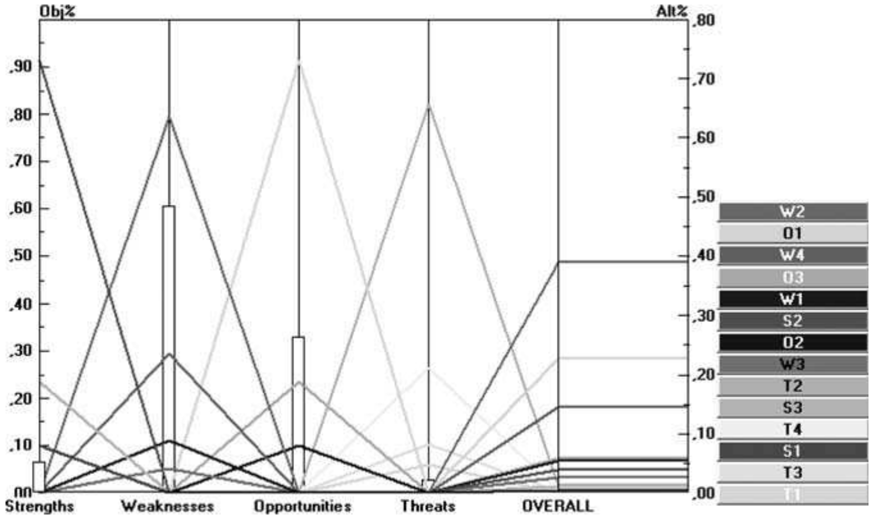

Figure 7. Sensitivity with respect to the goal: SWOT groups

4.5.2 Fuzzy SWOT-AHP Analysis

Using Chang's (1992) extent analysis, we obtained one eigenvector for the SWOT factors and four eigenvectors for the subfactors of SWOT. In the following discussion, the pair-wise comparison matrix among SWOT groups and one sample of the pair-wise comparisons for subfactors of SWOT groups are given (Tables 8 and 9). Some of the pair-wise comparisons are given in the appendix. The eigenvectors of all pair-wise comparisons among SWOT groups and subfactors of SWOT can be seen in Table 7.

From Table 7,

$$S_S = (2.90,3.67,4,67) \otimes (1/22.50,1/18,1/14.20) = (0.129,0.204,0.329)$$
$$S_W = (3.40,4.50,5.67) \otimes (1/22.50,1/18,1/14.20) = (0.151,0.250,0.399)$$
$$S_O = (5.50,7.00,8.50) \otimes (1/22.50,1/18,1/14.20) = (0.244,0.389,0.599)$$
and
$$S_T = (2.40,2.83,3.67) \otimes (1/22.50,1/18,1/14.20) = (0.107,0.157,0.258)$$

are obtained. Using these vectors, $V(S_W \geq S_S) = 1.00$ and other V values are obtained as 1.00, 1.00, 0.74, 0.79, 1.00, 1.00, 0.54, 0.31, 0.53, 1.00, and 0.06, respectively. Thus, the weight vector from Table 8 is calculated as $W_G = (0.165,0.278,0.528,0.030)^T$.

Table 7. The Fuzzy Pair-wise Comparison Matrix of the Weakness Criteria

W	W1	W2	W3	W4
W1	(1, 1, 1)	(2/5, 1/2, 2/3)	(1/2, 1, 3/2)	(2/5, 1/2, 2/3)
W2	(3/2, 2, 5/2)	(1, 1, 1)	(3/2, 2, 5/2)	(3/2, 2, 5/2)
W3	(2/3, 1, 2)	(2/5, 1/2, 2/3)	(1, 1, 1)	(2/5, 1/2, 2/3)
W4	(3/2, 2, 5/2)	(2/5, 1/2, 2/3)	(3/2, 2, 5/2)	(1, 1, 1)

Table 8. The priority weights of SWOT groups and factors

SWOT group	Priority of the group	SWOT factors	Priority of the factor within the group	Overall priority of the factor
Strengths	0.165	*S1. Formation of supervisory and executive committees*	0.083	0.0137
		S2. e-Transformation projects		
		S3. Support from top-level	0.764	0.1261
		management of public and private sectors	0.153	0.0252
Weaknesses	0.278	*W1. Lack of access to Internet among certain sections of the population*	0.052	0.0145
		W2. Lack of finance for capital investment in new technologies	0.487	0.1354
		W3. Need to change individual attitudes and organizational cultures.	0.105	0.0292
		W4. Poor economic power of citizens and businesses	0.356	0.0990
Opportunities	0.528	*O1. A candidate country from the European Union's information society perspective*	0.771	0.4071
		O2. Efficiency by reducing costs and layers of organizational processes by re-engineering	0.038	0.0201
		O3. New and improved services,	0.191	0.1008
Threats	0.03	*T1. Decentralized Internet governance,*	0.061	0.0018
		T2. Inadequate government IT security	0.565	0.0170
		T3. Copyright lawsuits	0.079	0.0024
		T4. Availability of "dangerous" information	0.295	0.0089

The weight vector from Table 9 is calculated as $W_W = (0.05, 0.49, 0.11, 0.36)^T$.

Table 9. The priority weights of SWOT groups and factors

With respect to S1	A1	A2	A3	A4	Inconsistency ratio	Priorities of alternatives with respect to S1
A1	1	1/5	6	5		0.352
A2	5	1	8	9		1.000
A3	1/6	1/8	1	1	0.07	0.084
A4	1/5	1/9	1	1		0.083
With respect to S2	A1	A2	A3	A4	Inconsistency ratio	Priorities of alternatives with respect to S2
A1	1	3	1/6	3		0.303
A2	1/3	1	1/4	3		0.188
A3	6	4	1	8	0.10	1.000
A4	1/3	1/3	1/8	1		0.088
With respect to S3	A1	A2	A3	A4	Inconsistency ratio	Priorities of alternatives with respect to S3
A1	1	1/3	6	8		0.579
A2	3	1	5	9		1.000
A3	1/6	1/5	1	2	0.07	0.144
A4	1/8	1/9	1/2	1		0.080

Figure 8 illustrates the priority weights of the categorized subfactors whose numerical values are given in Table 10.

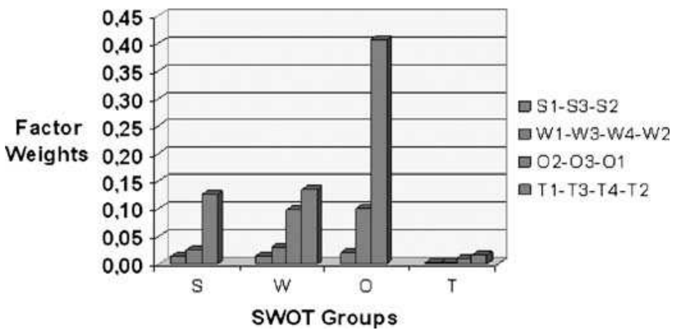

Figure 8. The priority weights of the categorized subfactors

5. POSSIBLE E-GOVERNMENT STRATEGIES

Possible e-government strategies of Turkey are the alternatives for the AHP problem above. The e-government initiatives will deliver more and better services to citizens through more effective inter- and intra-governmental teamwork. The government will deliver these services at a lower cost through the reduction of redundant systems and applications. The Office of E-Government and IT will pursue the following strategies to accomplish the goal:

(A1) Simplify work processes to improve service to citizens. The individual e-government projects will be driving the migration of systems, data, and processes to a common solution that better meets citizen needs. Agencies will be setting up solutions that cross traditional organization "silos," based on e-business principles of "buy once, use many" and "collect once, use many."

(A2) Use the annual budget process and other requirements to support e-government implementation. There will be a continued consolidation of work plans and investments in technologies acquired by different agencies for like purposes and external-facing transactions platforms. Agencies will continue to reduce redundant spending and improve the return on IT investments through the use of business cases, capital planning, investment and control process, and through other means, such as enterprise licensing.

(A3) Improve project delivery through development, recruitment, and retention of a qualified IT workforce. The Turkish Government will support the efforts to analyze information resource management and personnel needs and assess and upgrade current IT training programs.

(A4) Continue to modernize agency IT management around citizen-centered lines of business. The next series of e-government initiatives will drive improvement in the way the Turkish Government makes and monitors IT investments. The Administration has defined an annual cycle for identifying, analyzing, and deploying opportunities to integrate and consolidate activities along business lines that cross agency boundaries. The policy of the Administration is that IT transformation will be based on consolidation along lines of business and citizen needs: Agencies will have to make the business case for developing a unique solution.

6. EVALUATION OF E-GOVERNMENT STRATEGIES IN TURKEY

In this section the importance weights of the e-government strategies are determined to find the strategy with the largest weight that should be implemented first.

6.1 Crisp AHP

Table 9 gives the pair-wise comparison matrices of alternative strategies with respect to strengths together with the inconsistency ratios. Some other pair-wise comparison matrices of alternative strategies with respect to weaknesses, opportunities, and threats are given in the Appendix.

Using the Expert Choice software, we obtained the results shown in Figure 9. The rank order of the e-government strategies is *A1-A2-A3-A4.*

In Figures 10 and 11, a sensitivity analysis is given for SWOT groups and strategy alternatives

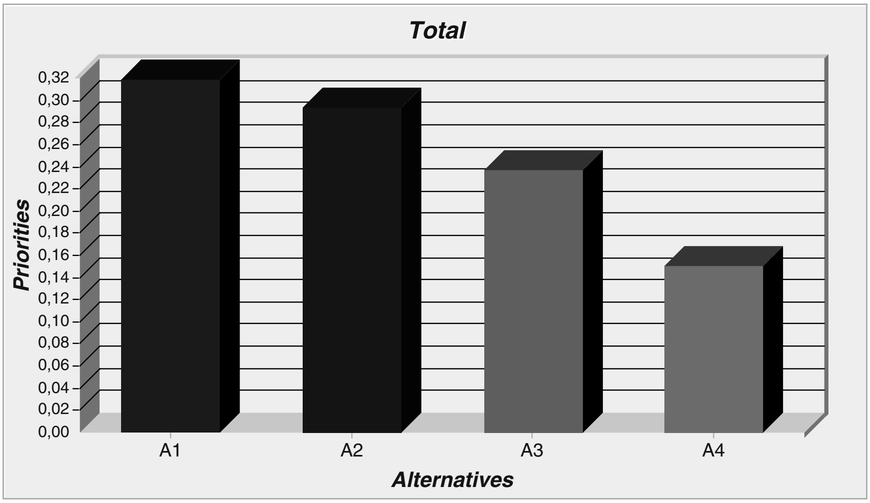

Figure 9. Priorities of e-government strategies

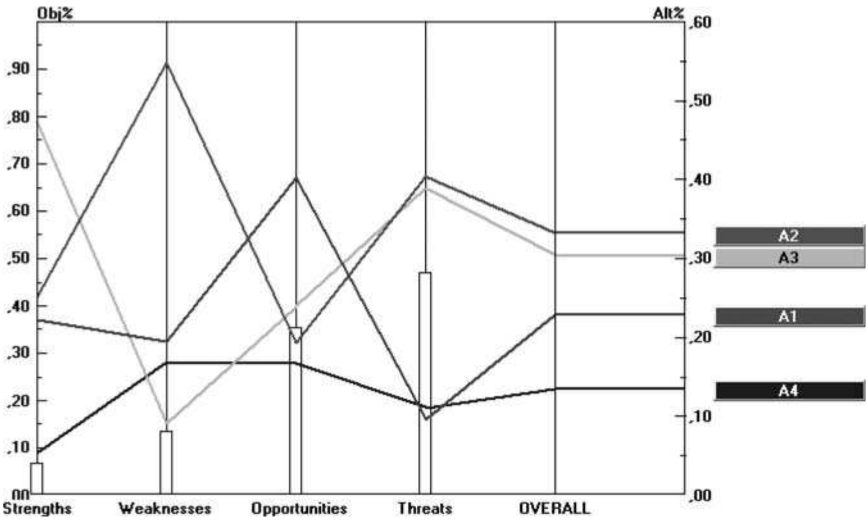

Figure 10. Sensitivity analysis for SWOT groups and strategy alternatives

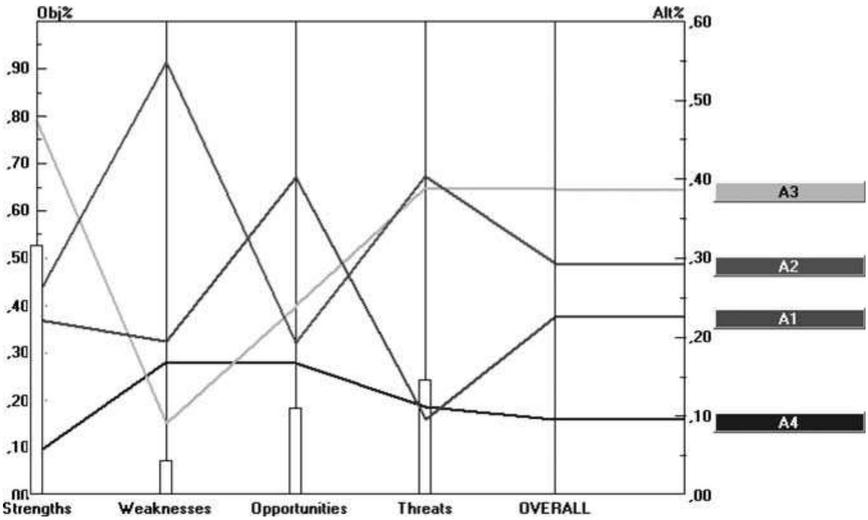

Figure 11. Sensitivity analysis for SWOT groups and strategy alternatives

From Figure 10, we see the overall weights on the right side of the figure, which indicate that *A2* (*Use the annual budget process and other requirements to support e-government implementation*) is the most important strategy of all. When we increase the weight of the *threats* group to make it the largest of all the groups, as illustrated on the *threats* line, the rank order is *A2-A3-A1-A4*. From Figure 11, we see the overall weights on the right side of the figure, which indicate that A3 (*Improve project delivery through development, recruitment and retention of a qualified IT workforce*) is the most important strategy of all. When we increase the weight of the *strengths* group to make it the largest of all the groups, as illustrated on the *strengths* line, the rank order is *A3-A2-A1-A4*.

6.2 Fuzzy AHP

A sample pair-wise comparison matrix of alternative strategies with respect to subfactors is given in Tables 10 and 11. Based on Chang's (1992) extent analysis, 14 eigenvectors for the e-government strategies with respect to the subfactors are obtained and given in Table 11 and the overall result is given in Table 12.

Table 10. The Fuzzy Evaluation of Alternatives with Respect to the Sub-Attribute O1

O1	A1	A2	A3	A4
A1	(1, 1, 1)	(1, 3/2, 2)	(3/2, 2, 5/2)	(2, 5/2, 3)
A2	(1/2, 2/3, 1)	(1, 1, 1)	(1, 3/2, 2)	(1, 3/2, 2)
A3	(2/5, 1/2, 2/3)	(1/2, 2/3, 1)	(1, 1, 1)	(1, 3/2, 2)
A4	(1/3, 2/5, 1/2)	(1/2, 2/3, 1)	(1/2, 2/3, 1)	(1, 1, 1)

Table 11. Weights of Attributes and Scores of Alternatives—Summary

	S			W				O			T			
	0.165			0.278				0.528			0.030			
	S1	S2	S3	W1	W2	W3	W4	O1	O2	O3	T1	T2	T3	T4
	0.083	0.764	0.153	0.052	0.487	0.105	0.356	0.771	0.038	0.191	0.061	0.565	0.079	0.295
A1	0.32	0.16	0.24	0.42	0.25	0.00	0.00	0.52	0.56	0.00	0.56	0.00	0.03	0.15
A2	0.68	0.17	0.71	0.13	0.68	0.05	0.61	0.30	0.05	0.00	0.34	0.64	0.15	0.00
A3	0.00	0.54	0.05	0.09	0.00	0.39	0.00	0.17	0.00	0.61	0.04	0.36	0.71	0.52
A4	0.00	0.12	0.00	0.36	0.07	0.56	0.39	0.01	0.39	0.39	0.06	0.00	0.11	0.32

The weight vector from Table 12 is calculated as $W_{AO1} = (0.52, 0.30, 0.17, 0.01)^T$.

Table 12. Results of Fuzzy AHP

	Priority Weights
A2	0.341
A1	0.298
A3	0.223
A4	0.139

7. CONCLUSION

Electronic government is no longer just an option but a necessity for countries aiming for better governance. People and policies play the primary role in making e-government a success. The framework explained in this chapter provides a direction for consideration of the evaluation of e-government strategies. The case study of Turkey provides an illustrative reference for the strategy evaluation. This model would be beneficial for evaluating any other e-government strategies in the country and for comparing its priority with the other e-government strategies. The selection of various SWOT factors depends on the system profile, the type of services being offered, and the profile of the citizen being served. The qualitative analysis of these factors and strategies is highly subjective and may differ from one expert to another.

Two different SWOT-AHP approaches (the crisp and the fuzzy cases) to the e-government strategy selection produced different rankings but close priority weights. That was caused by the type of information gathered from the experts. In the first method, they are asked to agree on precise values about alternative strategies, SWOT factors, and subfactors, whereas in the second what they needed to do was to agree on linguistic terms, which express their perceptions on alternative strategies, SWOT factors, and subfactors. The strategies *"simplify work processes to improve services to citizens"* and *"use the annual budget process and other requirements to support E-Government implementation"* have been found to be the two most important strategies for e-government in Turkey by both methods. New strategies may be proposed and added to the SWOT-AHP analysis. For additional research, the combination of SWOT and AHP may be changed to compare the results of this work with the ones of SWOT-TOPSIS, SWOT-Scoring, or SWOT-ELECTRE.

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APPENDIX

Some sample pair-wise comparison matrices for each level of the hierarchy are given as follows:

Table A.1. Crisp Pair-wise Comparison Matrix of the Opportunities

With respect to opportunities group	O1	O2	O3
O1	1	7	5
O2	1/7	1	1/3
O3	1/5	3	1

Table A.2. Crisp Pair-wise Comparison Matrix of the Threats

With respect to threats group	T1	T2	T3	T4
T1	1	1/9	1/2	1/6
T2	9	1	7	5
T3	2	1/7	1	1/3
T4	6	1/5	3	1

Table A.3. The Fuzzy Pair-wise Comparison Matrix of the Opportunities

With respect to GOAL	O1	O2	O3
O1	(1, 1, 1)	(2, 5/2, 3)	(3/2, 2, 5/2)
O2	(1/3, 2/5, 1/2)	(1, 1, 1)	(1/2, 1, 3/2)
O3	(2/5, 1/2, 2/3)	(2/3, 1, 2)	(1, 1, 1)

Table A.4. The Fuzzy Pair-wise Comparison Matrix of the Threats

With respect to GOAL	T1	T2	T3	T4
T1	(1, 1, 1)	(2/5, 1/2, 2/3)	(1/2, 1, 3/2)	(1/2, 2/3, 1)
T2	(3/2, 2, 5/2)	(1, 1, 1)	(3/2, 2, 5/2)	(2, 5/2, 3)
T3	(2/3, 1, 2)	(2/5, 1/2, 2/3)	(1, 1, 1)	(2/5, 1/2, 2/3)
T4	(1, 3/2, 2)	(1/3, 2/5, 1/2)	(3/2, 2, 5/2)	(1, 1, 1)

Table A.5. The Pair-wise Comparisons of Alternative Strategies with Respect to the Opportunities

With respect to O1	A1	A2	A3	A4	Inconsistency ratio	Normalized priorities of alternatives with respect to O1
A1	1	2	3	4	0.02	1.000
A2	1/2	1	1	3		0.500
A3	1/3	1	1	2		0.408
A4	1/4	1/3	1/2	1		0.204
With respect to O2	**A1**	**A2**	**A3**	**A4**	**Inconsistency ratio**	**Normalized priorities of alternatives with respect to O2**
A1	1	8	5	2	0.08	1.000
A2	1/8	1	2	1/7		0.145
A3	1/5	1/2	1	1/7		0.118
A4	1/2	7	7	1		0.729
With respect to O3	**A1**	**A2**	**A3**	**A4**	**Inconsistency ratio**	**Normalized priorities of alternatives with respect to O3**
A1	1	1	1/7	1/7	0.03	0.122
A2	1	1	1/6	1/7		0.128
A3	7	6	1	2		1.000
A4	7	7	1/2	1		0.730

Table A.6. The Pair-wise Comparisons of Alternative Strategies with Respect to Threats

With respect to T1	A1	A2	A3	A4	Inconsistency ratio	Normalized priorities of alternatives with respect to T1
A1	1	2	3	4	0.03	1.000
A2	1/2	1	2	4		0.647
A3	1/3	1/2	1	1		0.288
A4	1/4	1/4	1	1		0.228
With respect to T2	**A1**	**A2**	**A3**	**A4**	**Inconsistency ratio**	**Normalized priorities of alternatives with respect to T2**
A1	1	1/9	1/7	1	0.01	0.106
A2	9	1	2	7		1.000
A3	7	1/2	1	5		0.611
A4	1	1/7	1/5	1		0.122

With respect to T3	A1	A2	A3	A4	Inconsistency ratio	Normalized priorities of alternatives with respect to T3
A1	1	1/2	1/5	1/2	0.03	0.151
A2	2	1	1/4	2		0.317
A3	5	4	1	5		1.000
A4	2	1/2	1/5	1		0.214
With respect to T4	**A1**	**A2**	**A3**	**A4**	**Inconsistency ratio**	**Normalized priorities of alternatives with respect to T4**
A1	1	3	1/6	1/2	0.08	0.222
A2	1/3	1	1/5	1/6		0.108
A3	6	5	1	3		1.000
A4	2	6	1/3	1		0.445

Table A.7. The Fuzzy Pair-wise Comparisons of Alternatives with Respect to Opportunities

With respect to O2	A1	A2	A3	A4
A1	(1, 1, 1)	(2, 5/2, 3)	(2, 5/2, 3)	(1, 3/2, 2)
A2	(1/3, 2/5, 1/2)	(1, 1, 1)	(2/3, 1, 2)	(2/5, 1/2, 2/3)
A3	(1/3, 2/5, 1/2)	(1/2, 1, 3/2)	(1, 1, 1)	(2/5, 1/2, 2/3)
A4	(1/2, 2/3, 1)	(3/2, 2, 5/2)	(3/2, 2, 5/2)	(1, 1, 1)
With respect to O3	**A1**	**A2**	**A3**	**A4**
A1	(1, 1, 1)	(1, 1, 1)	(1/2, 2/3, 1)	(2/5, 1/2, 2/3)
A2	(1, 1, 1)	(1, 1, 1)	(1/2, 2/3, 1)	(2/5, 1/2, 2/3)
A3	(2, 5/2, 3)	(2, 5/2, 3)	(1, 1, 1)	(1, 3/2, 2)
A4	(3/2, 2, 5/2)	(3/2, 2, 5/2)	(1/2, 2/3, 1)	(1, 1, 1)

Table A.8. The Fuzzy Pair-wise Comparisons of Alternatives with Respect to Threats

With respect to T1	A1	A2	A3	A4
A1	(1, 1, 1)	(1, 3/2, 2)	(2, 5/2, 3)	(2, 5/2, 3)
A2	(1/2, 2/3, 1)	(1, 1, 1)	(1, 3/2, 2)	(3/2, 2, 5/2)
A3	(1/3, 2/5, 1/2)	(1/2, 2/3, 1)	(1, 1, 1)	(1/2, 1, 3/2)
A4	(1/3, 2/5, 1/2)	(2/5, 1/2, 2/3)	(2/3, 1, 2)	(1, 1, 1)
With respect to T2	**A1**	**A2**	**A3**	**A4**
A1	(1, 1, 1)	(2/7, 1/3, 2/5)	(1/3, 2/5, 1/2)	(2/3, 1, 2)
A2	(5/2, 3, 7/2)	(1, 1, 1)	(3/2, 2, 5/2)	(5/2, 3, 7/2)
A3	(2, 5/2, 3)	(2/5, 1/2, 2/3)	(1, 1, 1)	(2, 5/2, 3)
A4	(1/2, 1, 3/2)	(2/7, 1/3, 2/5)	(1/3, 2/5, 1/2)	(1, 1, 1)

With respect to T3	A1	A2	A3	A4
A1	(1, 1, 1)	(1/2, 2/3, 1)	(2/7, 1/3, 2/5)	(2/3, 1, 2)
A2	(1, 3/2, 2)	(1, 1, 1)	(2/5, 1/2, 2/3)	(1/2, 1, 3/2)
A3	(5/2, 3, 7/2)	(3/2, 2, 5/2)	(1, 1, 1)	(2, 5/2, 3)
A4	(1/2, 1, 3/2)	(2/3, 1, 2)	(1/3, 2/5, 1/2)	(1, 1, 1)
With respect to T4	A1	A2	A3	A4
A1	(1, 1, 1)	(1, 3/2, 2)	(2/5, 1/2, 2/3)	(1/2, 2/3, 1)
A2	(1/2, 2/3, 1)	(1, 1, 1)	(2/5, 1/2, 2/3)	(2/5, 1/2, 2/3)
A3	(3/2, 2, 5/2)	(3/2, 2, 5/2)	(1, 1, 1)	(3/2, 2, 5/2)
A4	(1, 3/2, 2)	(3/2, 2, 5/2)	(2/5, 1/2, 2/3)	(1, 1, 1)

FUZZY OUTRANKING METHODS: RECENT DEVELOPMENTS

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Abstract: The main objective of this chapter is to account for the most recent developments related to fuzzy outranking methods with a particular focus on the fuzzy outranking method developed by the authors. The valued outranking methods PROMETHEE and ELECTRE III, which are the outranking methods the most used for application in real-life multi-criteria decision aid problems, are also presented. The description of the general outranking approach is provided.

Key words: Outranking method, fuzzy outranking relation, pair-wise comparison, multicriteria decision aid

1. INTRODUCTION

Outranking methods form one of the main families of methods in multi-criteria decision aid (MCDA). Other important methods are multi-attribute utility theory (MAUT) methods, interactive methods, and the analytic hierarchy process (AHP).

It is worth recalling that the first outranking method called ELECTRE I was developed by Bernard Roy and published in 1968. Since then, a series of outranking methods were developed mainly during the 1970s and 1980s. Among them we can quote ELECTRE II (Roy and Bertier, 1973), ELECTRE III (Roy, 1978), QUALIFLEX (Paelinck, 1978), ORESTE (Roubens, 1982; Pastijn and Leysen, 1989), ELECTRE IV (Roy and Hugonnard, 1982), MELCHIOR (Leclercq, 1984), PROMETHEE I and II

C. Kahraman (ed.), Fuzzy Multi-Criteria Decision Making.

(Brans and Vincke, 1985), TACTIC (Vansnick, 1986), MAPPACC (Matarazzo, 1986), and PRAGMA (Matarazzo, 1986).

The outranking methods are based on the construction and the exploitation of an outranking relation. The underlying idea consists of accepting a result less rich than the one yielded by multi-attribute utility theory by avoiding the introduction of mathematical hypotheses that are too strong and asking the decision maker some questions that are too intricate (Vincke, 1992a). The concept of an outranking relation is introduced by Bernard Roy who is the founder of outranking methods. According to Roy (1974), an outranking relation is a binary relation S defined on the set of alternatives $\boldsymbol{A}$ such that for any pair of alternatives $(a,b) \in \boldsymbol{A}\times\boldsymbol{A}$: aSb if, given what is known about the preferences of the decision maker, the quality of the evaluations of the alternatives and the nature of the problem under consideration, there are sufficient arguments to state that the alternative a is at least as good as the alternative b, while at the same time no strong reason exists to refuse this statement.

In contrast to the other methods, the outranking methods have the characteristic of allowing incomparability between alternatives. This characteristic is important in situations where some alternatives cannot be compared for one or another reason. According to Siskos (1982), incomparability between two alternatives can occur because of a lack of information, inability of the decision maker to compare the two alternatives, or his refusal to compare them (Siskos, 1982).

In contrast to the valued outranking methods that are well documented in the literature and have been intensively used in practice since 1978 with the publication of ELECTRE III, the fuzzy outranking methods are very recent and are not well documented in the literature, and this is one of the motivations for the redaction of this chapter.

The chapter is structured as follows. The main elements of a general outranking approach are described in Section 2. Section 3 is devoted to the presentation of the PROMETHEE and ELECTRE III, which are the main valued outranking methods considered in both theory and applications. The fuzzy outranking methods are presented in Section 4. Some concluding remarks are given in Section 5.

2. THE OUTRANKING APPROACH

An outranking method is applicable for MCDA problems where the elements of a finite set of alternatives $\boldsymbol{A} = \{a_1, a_2, \ldots, a_n\}$ have to be compared on the basis of the preferences of the decision maker regarding their performances with respect to the elements of a finite set of criteria $F = \{g_1, g_2, \ldots, g_m\}$. It is assumed that each alternative a_i, $i = 1, \ldots, n$ can be evaluated with respect to each criterion g_j, $j = 1, \ldots, m$. The evaluations can be quantitative or qualitative. They can also be deterministic or nondeterministic. In the nondeterministic case, they can be fuzzy or stochastic.

The objective of outranking methods is provide decision aid to decision makers in the form of a subset of "best" alternatives or a partial or complete ranking of alternatives (Pasche, 1991).

According to Roy (1991), the preferences in the outranking concept are determined at two different levels as follows:

- Level of preferences restricted to each criterion. For example, to each criterion g_j, it is possible to associate a restricted outranking relation S_j such that for any two alternatives a and b in $\boldsymbol{A}$:

$$aS_j b \Leftrightarrow a, \text{with respect to } g_j, \text{is at least as good as } b \qquad (1)$$

- Level of comprehensive preferences where all criteria are taken into account.

The meaning of an outranking relation is given in Section 1. However, there is a need for a set of conditions to recognize whether a given binary relation can be an outranking relation. The following definition is provided in (Perny and Roy, 1992)

DEFINITION 1.

A fuzzy relation S_j defined on A^2 is said to be a *monocriterion outranking index* for a criterion g_j if a real-valued function t_j, exists defined on A^2, verifying $S_j(a, b) = t_j(a_j, b_j)$ for all a and b in A with a_j and b_j being the crisp scores of a and b on criterion g_j such that:

- $\forall y_0 \in \Re, t_j(x, y_0)$ is a nondecreasing function of x,
- $\forall x_0 \in \Re, t_j(x_0, y)$ is a nonincreasing function of y,
- $\forall z \in \Re, t_j(z, z) = 1$.

It is worth noticing that the three conditions in this definition are also valid for fuzzy outranking relations constructed from fuzzy evaluations on criteria and for global outranking relations.

In the literature, confusion abounds regarding valued and fuzzy outranking relations, and they are often used interchangeably. Even if the valued and fuzzy outranking relations are similar from a mathematical point of view, they represent two different situations:

- The valued outranking relation represents a crisp situation, and the value $S(a, b) \in [0, 1]$ represents the intensity with which the alternative a outranks alternative b and $S(a, b)$ is constructed from crisp evaluations of alternatives a and b.
- The valued outranking relation represents a fuzzy situation, and the value $S(a,b) \in [0, 1]$ represents the degree with which the alternative a is R-related to b and $S(a,b)$ is constructed from fuzzy evaluations of alternatives a and b.

An outranking method is composed of two main phases that are the construction of a global outranking relation and the exploitation of this relation.

The construction phase is composed of two main steps:

- Construction of an outranking relation or related relations such as concordance and discordance indices with respect to each criterion,
- The aggregation of the single outranking relations into a global outranking relation.

The exploitation phase of a valued/fuzzy outranking method can be dealt with in three different ways (Fodor and Roubens, 1994):

- Transformation of the valued/fuzzy outranking relation into another valued/fuzzy relation having particular properties such as transitivity that are interesting for the ranking of alternatives,
- Determination of a crisp relation closed to the valued/fuzzy outranking relation and having specific properties,
- Use of a ranking procedure to obtain a score function as it is the case for PROMETHEE and ELECTRE III methods.

A detailed study of the exploitation phase in the case of crisp relations is provided in (Vincke, 1992b).

3. VALUED OUTRANKING METHODS

The outranking methods that are the most used for application in real-life MCDA problems are ELECTRE III and PROMETHEE, which are valued outranking methods since they are based on the construction and exploitation of a valued "outranking relation." ELECTRE stands for "ELimination Et Choix Traduisant la REalité," and PROMETHEE stands for "Preference Ranking Organization METHod for Enrichment Evaluations."

3.1 ELECTRE III

ELECTRE III is an outranking method proposed by Roy (1978) to deal with multi-criteria decision-making situations in which a finite set of alternatives should be ranked from the best to the worst. It is composed of the following steps:

- The construction of a valued outranking relation;
- The construction of two complete preorders based on descending and ascending distillation chains;
- The comparison of the two complete preorders in order to elaborate a final ranking of the alternatives. This comparison leads to a partial preorder in which it is possible that some alternatives are incomparable.

3.1.1 The Construction Phase of ELECTRE III

Let $\boldsymbol{A} = \{a_1, a_2, \ldots, a_n\}$ be a finite set of n alternatives and $F = \{g_1, g_2, \ldots, g_m\}$ a set of m criteria on which the alternatives in $\boldsymbol{A}$ will be evaluated. Without loss of generality, the criteria can be assumed to be maximizing, i.e., the higher the performance of an alternative on a criterion is, the better the alternative is. ELECTRE III is based on the definition of a valued outranking relation S such that for each ordered pair of alternatives (a,b), $S(a,b) \in [0, 1]$ represents the degree to which alternative a is at least as good as alternative b (the degree to which alternative a is not worse than alternative b).

3.1.1.1 Single Criterion Relations

With each criterion g_j ($j = 1, \ldots, m$) are associated four parameters: a weight w_j, a preference threshold p_j, an indifference threshold q_j, and a veto threshold v_j. It is naturally assumed that for each alternative a: $q_j(g_j(a)) \leq p_j(g_j(a)) \leq v_j(g_j(a))$.

With each criterion g_j ($j = 1, \ldots, m$) are associated a concordance index c_j and a discordance index d_j as follows which are shown in Figures 1 and 2 respectively.

$$c_j(a,b) = \begin{cases} 1 \text{ if } g_j(a) + q_j(g_j(a)) \geq g_j(b), \\ 0 \text{ if } g_j(a) + p_j(g_j(a)) \leq g_j(b), \\ \dfrac{p_j(g_j(a)) + g_j(a) - g_j(b)}{p_j(g_j(a)) - q_j(g_j(a))} \quad \text{otherwise}* \end{cases} \tag{2}$$

* may occur only in the case when $q_j(g_j(a)) \neq p_j(g_j(a))$.

$$d_j(a,b) = \begin{cases} 0 \;\; \textit{if} \;\; g_j(b) \leq g_j(a) + p_j(g_j(a)), \\ 1 \;\; \textit{if} \;\; g_j(b) \geq g_j(a) + v_j(g_j(a)), \\ \dfrac{g_j(b) - g_j(a) - p_j(g_j(a))}{v_j(g_j(a)) - p_j(g_j(a))} \quad \text{otherwise}* \end{cases} \tag{3}$$

* may occur only in the case when $p_j(g_j(a)) \neq v_j(g_j(a))$.

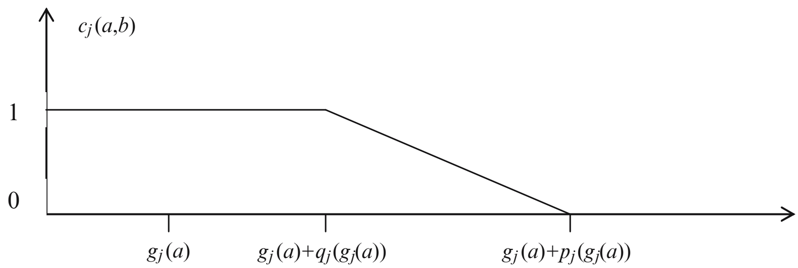


Figure 1. Concordance index of g_j

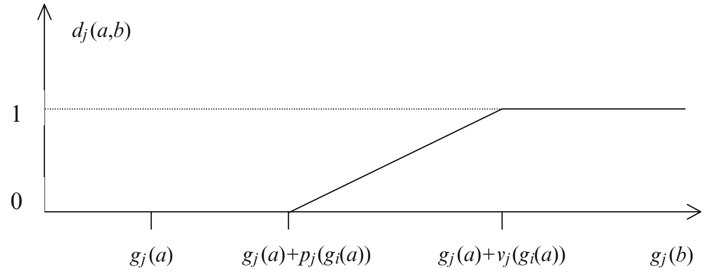


Figure 2. Discordance index of g_j

3.1.1.2 Global Valued Outranking Relation

For each ordered pair of alternatives (a,b), a concordance index $c(a,b)$ is computed in the following way:

$$c(a,b)=\frac{1}{W}\sum_{j=1}^{m} w_j c_j(a,b), \text{ where } W=\sum_{j=1}^{m} w_j \tag{4}$$

It is worth noticing that $c(a,b) = 1$ means that there is no criterion for which alternative b is better than alternative a and $c(a,b) = 0$ means that alternative a is worse than alternative b for all criteria.

The valued outranking relation S is constructed from the concordance and discordance indices. For each ordered pair of alternatives $(a,b) \in \boldsymbol{A}\times\boldsymbol{A}$, $S(a,b)$ is defined in the following way:

$$S(a,b)=\begin{cases} c(a,b) \text{ if } d_j(a,b)\le c(a,b), \ \forall j=1, \ldots, m \\ c(a,b)\times \prod\limits_{j\in J(a,b)} \dfrac{1-d_j(a,b)}{1-c(a,b)} \text{ otherwise} \end{cases} \tag{5}$$

where $J = \{j \in \{1, \ldots, m\} / d_j(a, b) > c(a, b)\}$.

The degree of outranking is equal to the concordance index when no criterion is discordant. When at least one criterion is discordant, the degree of outranking is equal to the concordance index multiplied by a factor lowering the concordance index in function of the importance of the discordances. At the extreme, when $d_j(a,b) = 1$ for some criterion g_j, $S(a,b) = 0$. Thus, for each ordered pair of alternatives $(a,b) \in \boldsymbol{A} \times \boldsymbol{A}$, $0 \le S(a,b) \le 1$. S is a valued outranking relation.

3.1.1.3 The Exploitation Phase of ELECTRE III

The second step in ELECTRE III consists in defining two complete preorders from the descending and the ascending distillation chains.

Let $\lambda_0 = \max_{a,b \in A} S(a,b)$. At each iteration of the descending or ascending distillation chain, a discrimination threshold $s(\lambda)$ and a crisp relation D are defined such that:

$$D(a,b) = \begin{cases} 1 \text{ if } S(a,b) > \lambda - S(\lambda) \\ 0 \text{ otherwise} \end{cases} \tag{6}$$

For each alternative a, a qualification score $Q(a)$ is computed as the number of alternatives that are outranked by a (number of alternatives b such that $D(a,b) = 1$) minus the number of alternatives, which outrank a (number of alternatives b such that $D(b,a) = 1$).

ELECTRE III provides the decision makers with two complete preorders. The first preorder is obtained in a descending manner starting with the selection of the alternatives with the best qualification score and finishing with the selection of the alternatives having the worst qualification score. The second preorder is obtained in an ascending manner, first selecting the alternatives with the worst qualification score and finishing with the assignment of the alternatives that have the worst qualification score.

3.1.1.4 Descending Distillation Chain

In the descending procedure, the set of alternatives having the largest qualification score constitutes the first distillate and is denoted as D_1. If D_1 contains only one alternative, the previous procedure is performed in the set $A \backslash D_1$. Otherwise it is applied to D_1 and a distillate D_2 will be obtained. If D_2 is a singleton, then the procedure is applied in $D_1 \backslash D_2$ if it is not empty; otherwise the procedure is applied in D_2. This procedure is repeated until the distillate D_1 is completely explored. Then, the procedure starts exploring $A \backslash D_1$ in order to find a new distillate. The procedure is repeated until a complete preorder of the alternatives is obtained. This procedure is called the descending distillation chain because it starts with the alternatives having the highest qualification and ends with the alternatives having the lowest qualification.

The result of the descending procedure is a set of classes $\overline{C}_1, \overline{C}_2, \ldots, \overline{C}_k$ with $k \le n$. The alternatives belonging to the same class are considered to be ex-æquo (indifferent), and an alternative belonging to a class outranks all the alternatives belonging to classes with higher indices. Thus, a first complete preorder of the alternatives is obtained.

3.1.1.5 Ascending Distillation Chain

The ascending procedure is the same as the descending procedure except that the criterion of selecting the alternatives is based on the principle of the lowest qualification. The result of this procedure is a set of classes $\underline{C}_1$, $\underline{C}_2$, …, $\underline{C}_h$ with $h \leq n$. These classes are written in such a way that two alternatives in the same class are considered to be ex-æquo and an alternative belonging to a class outranks all the alternatives belonging to classes with lower indices. Thus, a second complete preorder of the alternatives is obtained.

3.1.1.6 Partial Preorder of ELECTRE III

The result of ELECTRE III is a partial preorder of the alternatives based on the comparison of the two complete preorders obtained by means of the descending and the ascending distillation chains.

3.1.2 Main Features of ELECTRE III

ELECTRE III has many interesting features among which we can quote:

- Handling imprecise and uncertain information about the evaluation of alternatives on criteria by using indifference and preference thresholds,
- Consideration of incomparability between alternatives; when two alternatives cannot be compared in terms of preference or indifference, they are considered to be incomparable. Indeed, sometimes the information available is insufficient to decide whether two alternatives are indifferent or one is preferred to the other,
- Use of veto thresholds. This is very important for some problems such as those involving environmental and social impacts assessment. According to Rogers and Bruen (1998), within an environmental assessment, it seems appropriate to define a veto as the point at which human reaction to the criterion difference becomes so adverse that it places an "environmental stop" on the option in question. The same can be said about social impact assessment.

ELECTRE III is widely used for different real-world applications such as environmental impact assessment and selection problems in various domains. Examples of these applications can be found in Augusto et al. (2005), Beccali et al. (1998), Bufardi et al. (2004), Cote and Waaub (2000), Hokkanen and Salminen (1994, 1997), Kangas et al. (2001), Karagiannidis and Moussiopoulos (1997), Maystre et al. (1994), Rogers and Bruen (2000), Roy et al. (1986), Teng and Tzeng (1994), and Tzeng and Tsaur (1997). The list is not exhaustive and is given just for illustrative purposes to show the varied and numerous applications of the ELECTRE III method.

3.1.3 Illustrative Example

This illustrative example is taken from Bufardi et al. (2004). The problem considered consists of selecting the best compromise end-of-life (EOL) alternative to treat a vacuum cleaner at its EOL. Theoretically the number of potential EOL alternatives that can be considered is very high. In general only a few EOL alternatives are interesting. Users have their own ways for defining EOL alternatives depending on activity, objectives, experience and constraints from market, legislation, and available technology. In this illustrative example, five EOL alternatives are considered and described as follows. EOL alternative 1 consists of recycling as much as possible and incinerating the rest. EOL alternative 2 consists of recycling only parts with benefits and incinerating the rest. EOL alternative 3 consists of recycling all metals that cannot be incinerated and incinerating all the rest. EOL alternative 4 consists of reusing the motor, recycling metals, and incinerating the rest. EOL alternative 5 consists of landfilling all. The five EOL alternatives are presented in Table 1. The criteria used for the evaluation of EOL alternatives are presented in Table 2. The detailed description of the environmental criteria presented in Table 2 can be found in Goedkoop and Spriensma (2000). Once the EOL alternatives

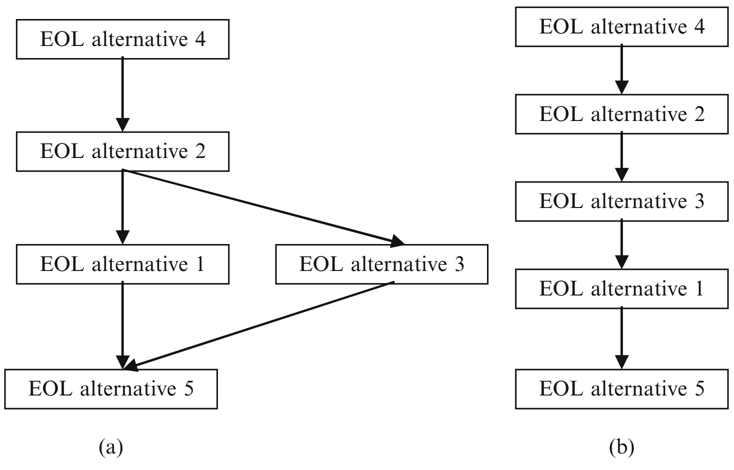


Figure 3. Partial and median preorder

and criteria are selected, each EOL alternative is evaluated with respect to each criterion as shown in Table 3. The results of applying ELECTRE III can be presented in the form of a partial preorder as shown in Figure 3a or a median preorder as shown in Figure 3(b).

Table 1. The EOL Alternatives

No.	Component/subassembly	EOL alternatives				
		1	2	3	4	5
1	Dust bin	*REC*	*INC*	*INC*	*INC*	*LND*
2	2 x Inner Cover	*REC*	*INC*	*INC*	*INC*	*LND*
3	Inner filter asb	*INC*	*INC*	*INC*	*INC*	*LND*
4	Dust bin cover	*INC*	*INC*	*INC*	*INC*	*LND*
5	Lock ring	*REC*	*INC*	*INC*	*INC*	*LND*
6	Spring	*REC*	*INC*	*REC*	*REC*	*LND*
7	Power button cover (+ button)	*REC*	*INC*	*INC*	*INC*	*LND*
8	Spring	*REC*	*INC*	*REC*	*REC*	*LND*
9	Upper VC case	*REC*	*INC*	*INC*	*INC*	*LND*
10	Suction tube	*REC*	*INC*	*INC*	*INC*	*LND*
11	Suction tube sealing	*INC*	*INC*	*INC*	*INC*	*LND*
12	Intermediate tube	*REC*	*INC*	*INC*	*INC*	*LND*
13	Cables	*REC*	*REC*	*INC*	*INC*	*LND*
14	Valve	*INC*	*INC*	*INC*	*INC*	*LND*
15	Intern sealing 1	*INC*	*INC*	*INC*	*INC*	*LND*
16	Intern sealing 2	*INC*	*INC*	*INC*	*INC*	*LND*
17	Spring	*REC*	*INC*	*REC*	*REC*	*LND*
18	Middle	*REC*	*INC*	*INC*	*INC*	*LND*
19	Hepa cover	*REC*	*INC*	*INC*	*INC*	*LND*
20	Hepa filter	*INC*	*INC*	*INC*	*INC*	*LND*
21	Cable coil cover	*REC*	*INC*	*INC*	*INC*	*LND*
22	Cable coil	*INC*	*INC*	*INC*	*INC*	*LND*
23	Cable	*REC*	*REC*	*INC*	*INC*	*LND*
24	Motor Lock ring	*REC*	*INC*	*INC*	*INC*	*LND*
25	Motor bottom seal	*INC*	*INC*	*INC*	*INC*	*LND*
26	Motor sealing	*INC*	*INC*	*INC*	*INC*	*LND*
27	Motor foam	*INC*	*INC*	*INC*	*INC*	*LND*
28	Motor	*REC*	*REC*	*REC*	*REM*	*LND*
29	Motor housing half 2	*REC*	*INC*	*INC*	*INC*	*LND*
30	Motor housing Filter	*INC*	*INC*	*INC*	*INC*	*LND*
31	Motor housing half 1	*REC*	*INC*	*INC*	*INC*	*LND*
32	32 Motor housing seal	*INC*	*INC*	*INC*	*INC*	*LND*
33	Wheels	*INC*	*INC*	*INC*	*INC*	*LND*
34	Lower VC case	*REC*	*INC*	*INC*	*INC*	*LND*
35	Spring	*REC*	*INC*	*REC*	*REC*	*LND*

* remanufacturing/reuse (REM), recycling (REC), incineration with energy recovery (INC), disposal to landfill (LND)

Table 2. List of Criteria

Category	Criterion	Unit	Direction of preferences
Economic	EOL Treatment Cost (C)	[CHF]	Minimization
	Human Health (HH)	[DALY]	Minimization
Environmental	Ecosystem Quality (EQ)	[PDF*m2yr]	Minimization
	Resources (R)	[MJ surplus]	Minimization

Table 3. Evaluation of EOL Alternatives

	Human health (HH) [DALY]	Ecosystem quality (EQ) [PDF*m2yr]	Resources (R) [MJ surplus]	EOL treatment cost (C) [CHF]
EOL alternative 1	−1.08E-05	−0.471	−18.1	0.644125
EOL alternative 2	−0.951E-05	−0.962	−7.49	−0.10601
EOL alternative 3	−0.724E-05	−0.896	−6.76	0.01108
EOL alternative 4	−2.90E-05	−2.02	−36.8	−4.86022
EOL alternative 5	0.0271E-05	0.0103	0.0101	0.38101

3.2 PROMETHEE

PROMETHEE is a MCDA method based on the construction and the exploitation of a valued outranking relation π (Brans and Vincke, 1985). Two complete preorders can be obtained by ranking the alternatives according to their incoming flow and their outgoing flow. The intersection of these two preorders yields the partial preorder of PROMETHEE I where incomparabilities are allowed. The ranking of the alternatives according to their net flow yields the complete preorder of PROMETHEE II.

3.2.1 The Construction Phase of PROMETHEE

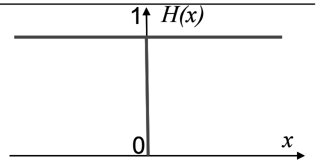
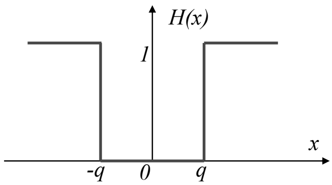
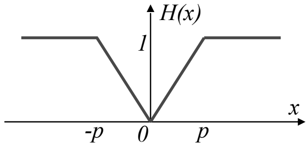
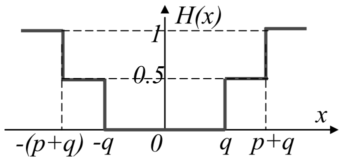
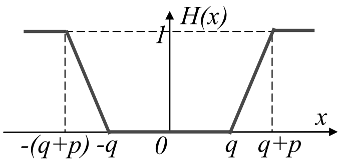
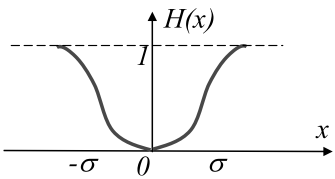
Let $\boldsymbol{A} = \{a_1, a_2, \ldots, a_n\}$ be a finite set of alternatives and $F = \{g_1, g_2, \ldots, g_m\}$ a finite set of criteria on which the alternatives will be evaluated. With each criterion g_j, $j = 1, 2, \ldots, m$, is assigned a weight p_j reflecting its relative importance.

For each pair of alternatives $(a,b) \in \boldsymbol{A} \times \boldsymbol{A}$, an outranking degree $\prod(a,b)$ is computed in the following way:

$$\Pi(a,b) = \frac{1}{P}\sum_{j=1}^{m} p_j H_j(a,b) \tag{7}$$

$P = \sum_{j=1}^{m} p_j$ and $H_j(a,b)$ are numbers between 0 and 1 that are a function of $g_j(a) - g_j(b)$. For the computation of $H_j(a,b)$'s, the decision maker is given six forms of curves described in Table 1. It is worth noticing that in Table 4, the six functions are described for a maximizing criterion where $H(x) = P(a,b)$ if $x \geq 0$ and $H(x) = P(b,a)$ if $x \leq 0$.

Table 4. List of Generalized Criteria

Type of criterion	Analytical definition	Shape
1. Usual	$H(x)=\begin{cases}0 & if\ x=0\\ 1 & if\ \lvert x\rvert>0\end{cases}$	
2. Quasi	$H(x)=\begin{cases}0 & if\ \lvert x\rvert\leq q\\ 1 & otherwise\end{cases}$	
3. Linear preference	$H(x)=\begin{cases}\lvert x\rvert/p & if\ \lvert x\rvert\leq p\\ 1 & otherwise\end{cases}$	
4. Level	$H(x)=\begin{cases}0 & \lvert x\rvert\leq q\\ 0.5 & q<\lvert x\rvert\leq p+q\\ 1 & \lvert x\rvert\geq q+p\end{cases}$	
5. Linear preference and indifference area	$H(x)=\begin{cases}0 & \lvert x\rvert\leq q\\ (\lvert x\rvert-q)/p & q<\lvert x\rvert\leq q+p\\ 1 & otherwise\end{cases}$	
6. Gaussian	$H(x)=\begin{cases}0 & x=0\\ 1-e^{-x^2/2\sigma^2} & \lvert x\rvert>0\end{cases}$	

3.2.2 The Exploitation Phase of PROMETHEE

With each alternative are associated two values $\phi^+(a)$ and $\phi^-(a)$.

- $\phi^+(a)$, which is called the *outgoing flow* and is computed in the following way:

$$\phi^+(a) = \sum_{b \in A} \Pi(a,b) \tag{8}$$

- $\phi^-(a)$, which is called the *incoming flow* and is computed in the following way:

$$\phi^-(a) = \sum_{b \in A} \Pi(b,a) \tag{9}$$

It is worth noticing that $\phi^+(a)$ represents the degree by which alternative *a* outranks the other alternatives and that $\phi^-(a)$ represents the degree by which alternative *a* is outranked by the other alternatives.

The higher the outgoing flow and the lower the incoming flow, the better the alternative. The two flows induce the following complete preorders (ranking of the alternatives with consideration of indifference) on the alternatives, where *P* and *I* are the preference relation and indifference relation, respectively:

- $aP^+b \Leftrightarrow \phi^+(a) > \phi^+(b)$
- $aI^+b \Leftrightarrow \phi^+(a) = \phi^+(b)$
- $aP^-b \Leftrightarrow \phi^-(a) < \phi^-(b)$
- $aI^-b \Leftrightarrow \phi^-(a) = \phi^-(b)$

where P^+, I^+ refer to the outgoing flows while P^-, I^- refer to the incoming flows.

By ranking the alternatives in the decreasing order of the numbers $\phi^+(a)$ and in the increasing order of the numbers $\phi^-(a)$, two complete preorders can be obtained. Their intersection yields the partial order of PROMETHEE I as follows:

$$aSb\,(a \text{ strictly outranks } b)\begin{cases}\text{if } aP^+b \text{ and } aP^-b\\ \text{or } aP^+b \text{ and } aI^-b\\ \text{or } aI^+b \text{ and } aP^-b\end{cases} \tag{10}$$

$$aIb \text{ (}a\text{ is indifferent to }b\text{) if } aI^{+}b \text{ and } aI^{-}b \tag{11}$$

$$aJb \text{ (}a\text{ and }b\text{ are incomparable) otherwise} \tag{12}$$

i.e., $\neg aSb, \neg bSa$ and $\neg aIb$, where "¬" denotes negation.

For each alternative a, a *net flow* $\phi(a)$ can be obtained by subtracting the incoming flow $\phi^{-}(a)$ from the outgoing flow $\phi^{+}(a)$; i.e., $\phi(a) = \phi^{+}(a) - \phi^{-}(a)$. By ranking the alternatives in the decreasing order of ϕ, one obtains the unique complete preorder of PROMETHEE II.

3.2.3 Main Features of PROMETHEE

PROMETHEE has many interesting features among which we can quote:

- It is easy to understand. The mathematical background behind PROMETHEE is not complicated and is easy to understand by the users. This is important for the transparency of the results,
- It is easy to use. For each criterion, the decision maker has to fix the weight of this criterion, and at most two parameters of the function are associated with the criterion in order to derive the single-valued outranking relation related to this criterion,
- Consideration of incomparability between alternatives through PROMETHEE I; when two alternatives cannot be compared in terms of preference or indifference, they are considered to be incomparable. Indeed, sometimes the information available is insufficient to decide whether two alternatives are indifferent or one is preferred to the other. PROMETHE is an outranking method easy to understand and to use.

That is why it is widely used for practical MCDA problems in various domains; see, e.g., Al-Rashdan et al. (1999), Anagnostopoulos et al. (2003), Babic and Plazibat (1998), Elevli and Demirci (2004), Geldermann et al. (2000), Gilliams et al. (2005), Goumas and Lygerou (2000), Hababou and Martel (1998), Kalogeras et al. (2005), Le Téno and Mareschal (1998), Mavrotas et al. (2006), and Petras (1997). The list is not exhaustive and is given just for illustrative purposes.

4. FUZZY OUTRANKING METHODS

In these methods, it is assumed that the evaluations of alternatives on criteria are fuzzy.

4.1 Fuzzy Outranking Method of Gheorghe et al.

The fuzzy outranking method presented in this subsection is published in Gheorghe et al. (2004, 2005). Full details can be found in Gheorghe (2005).

4.1.1 Construction of Monocriterion Fuzzy Outranking Relation

The construction of the monocriterion fuzzy outranking relation starts by analyzing the intervals, in our case, the α-cuts of fuzzy performance of two alternatives *a* and *b*.

Let us consider two normalized and convex fuzzy numbers *A* and *B*, representing the performances of alternatives *a* and *b*, respectively (Figure 4). Let μ_A and μ_B be the membership functions of *A* and *B,* respectively. Each α_i-cut is defined by the interval $(a_1^{\alpha_i}, a_2^{\alpha_i})$ for *A* and $(b_1^{\alpha_i}, b_2^{\alpha_i})$ for *B,* respectively, where $i = 1, \ldots, N,$ with *N* denoting the number of α-cuts considered.

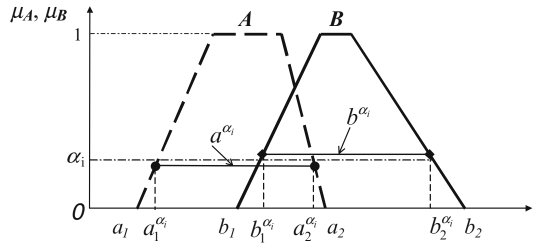


Figure 4. Fuzzy performances of alternatives *a* and *b*

The comparison performances of the alternatives *a* and *b* at the α_i-cut level using the mechanisms shown in Figures 5 and 6 are in accordance with common sense and represent two different view points. When the interval a^{α_i} is entirely on the left of the interval b^{α_i}, there is no doubt that *a* is worse than *b* and that the degree of trueness of the proposition "*a* is not worse than *b*" is 0. When starting to translate a^{α_i} to the right and the two intervals overlap, this degree of trueness increases and reaches the maximum value 1 at the moment when the lower limit (*left*) of a^{α_i} is equal with the lower limit (*left*) of b^{α_i} (Figure 5).

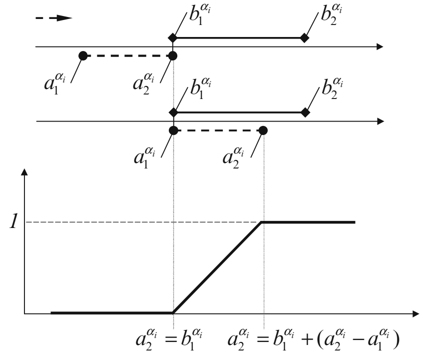


Figure 5. The first case of the achievement of a degree of trueness of 1 of the proposition "*a* is not worse than *b*"

A similar judgment can be performed for the case when the maximum degree of trueness is attained at the moment when the upper limit (*right*) of a^{α_i} is equal with the upper limit (*right*) of b^{α_i} (Figure 6).

Thus the reasoning we have done previously is suitable for the case when a higher value of performance is preferred to a lower value, in other words, for the case when we want to maximize the performance value with respect to a criterion. Similar reasoning can be followed for the case of a minimizing criterion.

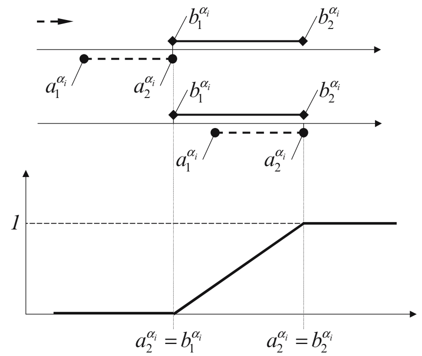


Figure 6. The second case of the achievement of a degree of trueness of 1 of the proposition "*a* is not worse than *b*"

DEFINITION 2.

For each α_i-cut level, two *left α_i-cut indices* are defined for, respectively, the case of maximizing and minimizing criteria as the functions $s_{l\_\max}^{\alpha_i}$ and, $s_{l\_\min}^{\alpha_i}$ from $I_{\Re} \times I_{\Re}$ to $[0,1]$, where $I_{\Re}$ is the set of all real intervals:

$$s_{l\_\max}^{\alpha_i}(a^{\alpha_i}, b^{\alpha_i}) = \begin{cases} 0, & a_2^{\alpha_i} < b_1^{\alpha_i} \\ \dfrac{a_2^{\alpha_i} - b_1^{\alpha_i}}{a_2^{\alpha_i} - a_1^{\alpha_i}}, & a_1^{\alpha_i} < b_1^{\alpha_i} \leq a_2^{\alpha_i} \\ 1, & a_1^{\alpha_i} \geq b_1^{\alpha_i} \end{cases} \tag{13}$$

$$s_{l\_\min}^{\alpha_i}(a^{\alpha_i}, b^{\alpha_i}) = \begin{cases} 0, & b_2^{\alpha_i} < a_1^{\alpha_i} \\ \dfrac{b_2^{\alpha_i} - a_1^{\alpha_i}}{b_2^{\alpha_i} - b_1^{\alpha_i}}, & b_1^{\alpha_i} < a_1^{\alpha_i} \leq b_2^{\alpha_i} \\ 1, & b_1^{\alpha_i} \geq a_1^{\alpha_i} \end{cases} \tag{14}$$

The *right α_i-cut indices* can be defined in a similar way as shown in the following definition.

DEFINITION 3.

For each α_i-cut level, two *right α_i-cut indices* are defined for, respectively, the case of maximizing and minimizing criteria as the functions $s_{r\_\max}^{\alpha_i}$ and $s_{r\_\min}^{\alpha_i}$ from $I_{\Re} \times I_{\Re}$ to $[0,1]$ such that:

$$s_{r\_\max}^{\alpha_i}(a^{\alpha_i}, b^{\alpha_i}) = \begin{cases} 0, & a_2^{\alpha_i} < b_1^{\alpha_i} \\ \dfrac{a_2^{\alpha_i} - b_1^{\alpha_i}}{b_2^{\alpha_i} - b_1^{\alpha_i}}, & b_1^{\alpha_i} \leq a_2^{\alpha_i} < b_2^{\alpha_i} \\ 1, & a_2^{\alpha_i} \geq b_2^{\alpha_i} \end{cases} \tag{15}$$

$$s_{r\_\min}^{\alpha_i}(a^{\alpha_i}, b^{\alpha_i}) = \begin{cases} 0, & b_2^{\alpha_i} < a_1^{\alpha_i} \\ \dfrac{b_2^{\alpha_i} - a_1^{\alpha_i}}{a_2^{\alpha_i} - a_1^{\alpha_i}}, & a_1^{\alpha_i} \leq b_2^{\alpha_i} < a_2^{\alpha_i} \\ 1, & b_2^{\alpha_i} \geq a_2^{\alpha_i} \end{cases} \tag{16}$$

DEFINITION 4.

For each α_i-cut level, two *right α_i-cut indices* are defined for, respectively the case of maximizing and minimizing criteria as the functions $s_{\max}^{\alpha_i}$ and, $s_{\min}^{\alpha_i}$ from $I_{\Re} \times I_{\Re}$ to $[0,1]$ such that:

$$s_{\max}^{\alpha_i}(a^{\alpha_i}, b^{\alpha_i}) = (1-\kappa)\cdot s_{l\_\max}^{\alpha_i}(a^{\alpha_i}, b^{\alpha_i}) + \kappa \cdot s_{r\_\max}^{\alpha_i}(a^{\alpha_i}, b^{\alpha_i}), \forall a,b \in A \quad (17)$$

$$s_{\min}^{\alpha_i}(a^{\alpha_i}, b^{\alpha_i}) = (1-\kappa)\cdot s_{r\_\min}^{\alpha_i}(a^{\alpha_i}, b^{\alpha_i}) + \kappa \cdot s_{l\_\min}^{\alpha_i}(a^{\alpha_i}, b^{\alpha_i}), \forall a,b \in A \quad (18)$$

The parameter $\kappa \in [0,1]$ represents the degree of optimism of the decision maker (Liou and Wang, 1992). It allows the decision maker to choose which side of the interval is more important. When *k* increases from 0 to 1, the degree of optimism increases, whereas the degree of pessimism decreases. This type of strategy will be called the horizontal strategy.

In the remaining of this chapter, the notation *s* or *S* are used without the index *min* or *max* and refer to the maximization case; however, the related statements are also valid for the minimization case, unless otherwise stated.

PROPOSITION 1.

The α_i-cut indices defined in Definition 4 are fuzzy outranking relations.

The transition from a fuzzy outranking relation defined at the α-cut level to a single criterion fuzzy outranking relation requires an aggregation procedure. Observing the case of fuzzy numbers *A* and *B* presented in Figure 7, it follows that the upper α-cut indices favor *B*, whereas the lower ones favor *A*. A compensative approach gives a certain discrimination power while still using the biggest amount of information contained in the fuzzy representation of the performances. This idea was exploited in area compensation methods for comparing fuzzy numbers by many authors (Chanas, 1987; Fortemps and Roubens, 1996; Matarazzo and Munda, 2001; Nakamura, 1986). The basic principle is that some nonintersecting areas (i.e., upper left and/or right external areas and lower left and/or right external areas in Figure 7) compensate each other. If we see the previously defined α-cut indices as relative intersections, then their aggregation can be seen as compensation between relative intersections, which is somehow related to the above-mentioned methods. If for linear membership functions the areas considered are relatively simple to be determined, for nonlinear cases, it becomes more difficult. In our α-cut approach besides the fact that we can use inputs stated as a set of α-cut intervals (which avoids possible necessary re-approximations of the original membership function), we prevent the use of integrals for calculating the areas used by an area compensation class of methods.

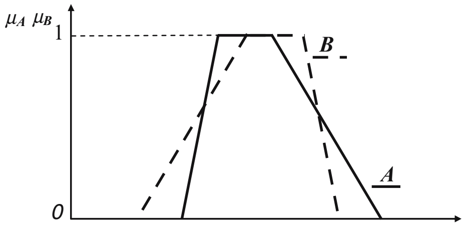


Figure 7. A complex case of comparison of fuzzy numbers

The function used to aggregate the α-cut indices is the *weighted root-power mean* defined for all *x* as follows (Smolíková and Wachowiak, 2002):

$$Fw_\lambda(x) = \left(\frac{\sum_{i=1}^{N} \omega_i (x_i)^\lambda}{\sum_{i=1}^{N} \omega_i} \right)^{\frac{1}{\lambda}} \tag{19}$$

Using the aggregation function Fw_λ to aggregate α_i-cut indices, $i = 1, \ldots, N$, we obtain single criterion fuzzy outranking relation *S* as follows:

$$S(A,B) = \left(\frac{\sum_{i=1}^{N} \omega_i \cdot (s^{\alpha_i}(a^{\alpha_i}, b^{\alpha_i}))^\lambda}{\sum_{i=1}^{N} \omega_i} \right)^{\frac{1}{\lambda}} \tag{20}$$

PROPOSITION 2.

The single criterion outranking index defined by the relation (20) satisfies the following properties:

- For any convex and normalized fuzzy number B^0, $S(A, B^0)$ is a non-decreasing function of *A*;

- For any convex and normalized fuzzy number A^0, $S(A^0, B)$ is a non-increasing function of B;
- For any fuzzy convex and normalized fuzzy number: $S(C, C) = 1$; hence S is reflexive.

Since the definitions of $s_l^{\alpha_i}$ and $s_r^{\alpha_i}$ allow them to take the value "0," at any α-cut level, some of the particular cases of the relation (20) are excluded:

- Geometrical mean for $\lambda = -1$ due to the possible division by zero;
- Product mean for $\lambda \rightarrow 0$, because of the risk of penalty of the result, when an α-cut level of S is 0.

As our intention is to offer the decision maker a flexible decision instrument, cases like *min* or *max* are also excluded. They are dictatorial aggregators, not allowing for compensation between lower and higher values.

Two particular cases are of special interest for the definition of the single criterion fuzzy outranking relation: the weighted arithmetic mean (21) and the weighted square average mean (22).

$$S(A,B) = \frac{\sum_{i=1}^{N} \omega_i \cdot s^{\alpha_i}(a^{\alpha_i}, b^{\alpha_i})}{\sum_{i=1}^{N} \omega_i} \tag{21}$$

$$S(A,B) = \left(\frac{\sum_{i=1}^{N} \omega_i \cdot (s^{\alpha_i}(a^{\alpha_i}, b^{\alpha_i}))^2}{\sum_{i=1}^{N} \omega_i} \right)^{\frac{1}{2}} \tag{22}$$

The consideration of weights for α-cut indices makes the final relation more flexible and offers to the decision maker the possibility to decide on the importance of the α-cut levels during the aggregation.

As we have to deal with an enlarged number of weights, equal with the number of α-cuts (which is N), we look to automatically generate the weights. We will search for a method that can give the possibility of changing the weighting vector, such that, for different personalities of the decision maker, we can build different weighting vectors. For example, in the case where the decision maker wants to rely his decisions on α-cuts with less uncertainty, he might be able to slide the highest weights to the highest α-cuts. Alternatively, one might want to give equal importance to all the α-cuts or to assign higher weights to lower α-cuts.

Here we will consider the case when the weights ω_i increase in a linear manner, so the interpolation of these points is a line. As we want to use the information given by all the α-cut indices, this kind of linearity looks convenient, because with two exceptions (the limit functions from this family, which will give 0 for the first α-cut, respectively for the last one), all the weights will be nonzero. The equation of such a line is:

$$\omega_i = \beta \cdot i + c \tag{23}$$

where β is the slope of the line and $c \in \Re$.

Through a series of calculations, using the three particular cases mentioned above and other conditions, the relation (23) becomes

$$\omega_i(\beta) = \beta \cdot \left(i - 1 - \frac{N-1}{2} \right) + \frac{1}{N} = \beta \cdot \left(i - \frac{N+1}{2} \right) + \frac{1}{N} \tag{24}$$

If we consider i as a continuous parameter, then ω_i transforms into a function $\omega(i, \beta) = \beta \times \left(i - 1 - \frac{N-1}{2} \right) + \frac{1}{N}$ of two variables, which can be represented as a surface, as shown in Figure 8.

Therefore, for the case of a maximizing criterion, we obtain the following the single criterion fuzzy outranking relation:

$$S_{\max}(A,B) = \sum_{i=1}^{N} \left(\beta \cdot \left(i - \frac{N+1}{2} \right) + \frac{1}{N} \right)$$

$$[(1-\kappa) \cdot s_{l\_\max}^{\alpha_i}(a^{\alpha_i}, b^{\alpha_i}) + \kappa \cdot s_{r\_\max}^{\alpha_i}(a^{\alpha_i}, b^{\alpha_i})]. \tag{25}$$

The expression of the single criterion fuzzy outranking relation for the case of a minimizing criterion can be obtained in a similar way.

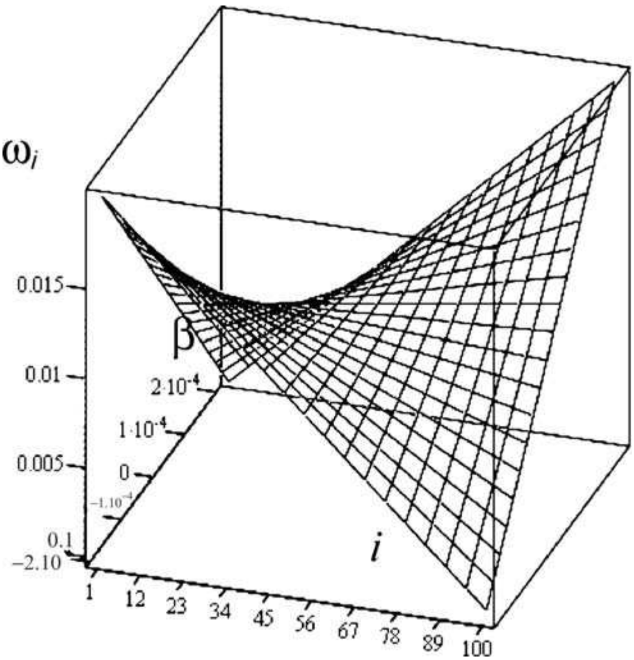


Figure 8. The function *ω(i,β)*

4.1.2 Aggregation of Single Fuzzy Outranking Relations

Here we are interested in aggregating over the set of criteria $g_1, \ldots, g_n$, the single criterion fuzzy outranking relations S_k into a global fuzzy outranking relation *S*.

Using an aggregation operator *M*, the global fuzzy outranking relation *S* is defined for each pair of alternatives (*a, b*) as follows:

$$S(a,b) = M\left[S_1(a,b),\ \ldots,\ S_n(a,b)\right]. \tag{26}$$

Obviously, *S* must have the properties of a fuzzy outranking relation, and the following proposition establishes the minimal conditions that an aggregator should fulfill in order to satisfy it.

PROPOSITION 3.

Any aggregator that satisfies the properties of idempotency and monotonicity with respect to the integrand, used to aggregate single criterion fuzzy outranking relations, leads to a global fuzzy relation that is a fuzzy outranking relation.

The Choquet integral (Grabisch, 1999; Marichal, 1999) is an aggregator that satisfies these two properties; consequently, the fuzzy relation obtained by aggregating single criterion fuzzy outranking relations through the use of a Choquet integral is a fuzzy outranking relation.

Considering the Choquet integral as the aggregation operator *M*, *S*(*A*, *B*) becomes

$$S(A,B) = \sum_{k=1}^{n} S_{(k)}(A,\ B)\ [\mu_{\{(k),\ldots,(n)\}} - \mu_{\{(k+1),\ldots,(n)\}}] \tag{27}$$

4.1.3 Exploitation of the Global Fuzzy Outranking Relation

The type of exploitation to be undergone by the global fuzzy outranking relation depends among others on the type of application for which this exploitation is to be used.

The problem for which this fuzzy outranking method was developed is one in which a large number of decisions has to be taken and for whose solving an automated decision-making procedure has to be put in place (e.g., the selection of the best EOL option for a large number of nodes in a disassembly tree of product with a complex assembly structure, the ranking of design concepts according to their lifecycle performance, including their EOL, etc.; see Gheorghe and Xirouchakis (2006) for a detailed description), but its application goes far beyond this context. It was shown that the α formulation (choice of the best alternative) of the exploitation problem (Roy, 1977) is the most suitable.

Roubens (1989) defined four generalized choice functions C_1, C_2, C_3, and C_4 with all of them being in the authors' opinion, intuitively attractive. As it can be seen from their definition, all these choice functions (and in general all possible choice functions) refer to the strength of the chosen alternative(s) over the rest of alternatives, so they measure somehow the domination of selected alternative(s) over the other or the nondomination of other alternatives on the selected one(s). The superscript "+" is used to denote the choice functions selecting the "best" alternative(s).

The *weak domination* of alternative *a* over all the other alternatives is defined as follows:

$$C_1^+(a) = \underset{b \in \boldsymbol{A} \setminus \{a\}}{T_1} S(a,b) \tag{28}$$

where T_1 is a *t*-norm and $S(a, b)$ is the degree, between 0 and 1, to which *a* is as good as *b*. To be in accordance with Orlovsky's (Orlovsky, 1978) reasoning and terminology, $C_1^+(a)$ can be interpreted as the degree of weak domination of *a* over all the other alternatives in ***A***. The choice set is given by

$$C_1^+(\boldsymbol{A},S) = \left\{ a \in \boldsymbol{A} \mid C_1^+(a) = \max_{b \in \boldsymbol{A}} C_1^+(b) \right\} \tag{29}$$

The *weak nondomination* of *a* by all the other alternatives is defined as follows:

$$C_2^+(a) = \underset{b \in \boldsymbol{A} \setminus \{a\}}{T_1} [1 - S(b,a)] \tag{30}$$

$C_2^+(a)$ is interpreted as the degree of weak nondomination of *a* by all the other alternatives in *A*. The choice set is given by

$$C_2^+(\boldsymbol{A},S) = \left\{ a \in \boldsymbol{A} \mid C_2^+(a) = \max_{b \in \boldsymbol{A}} C_2^+(b) \right\} \tag{31}$$

The *strict domination* of *a* over all the other alternatives is defined as follows:

$$C_3^+(a) = \underset{b \in \boldsymbol{A} \setminus \{a\}}{T_1} P(a,b) \tag{32}$$

$C_3^+(a)$ represents the degree of strict domination of *a* over *b*. *P* is the strict preference relation, and it is defined as $P(a, b)=T_2[(S(a, b), 1 - S(b, a)]$, with T_2 being a *t*-norm. The choice set is given by

$$C_3^+(\boldsymbol{A},S) = \left\{ a \in \boldsymbol{A} \mid C_3^+(a) = \max_{b \in \boldsymbol{A}} C_3^+(b) \right\} \tag{33}$$

The *strict nondomination* of *a* by all other alternatives is defined as follows:

$$C_4^+(a) = \underset{b \in \boldsymbol{A} \setminus \{a\}}{T_1} [1 - P(b,a)] \tag{34}$$

$C_4^+(a)$ represents the degree of strict nondomination of all the other alternatives on *a*. The choice set is given by

$$C_4^+(A,S) = \left\{ a \in A \mid C_4^+(a) = \max_{b \in A} C_4^+(b) \right\} \tag{35}$$

In contrast to the measurement of the strengths of alternatives, it is also interesting to measure their weaknesses. Four weakness-based choice functions C_5^-, C_6^-, C_7^-, and C_8^- are presented in the following.

- The *weak domination* of all alternatives on alternative *a*, representing the degree to which *a* is *weakly dominated* by all the other alternatives, is defined as follows:

$$C_5^-(a) = \underset{b \in \boldsymbol{A} \setminus \{a\}}{T_1} S(b,a) \tag{36}$$

The choice set corresponding to the weak domination (of all alternatives on a given alternative) function C_5^- is given by

$$C_5^-(\boldsymbol{A},S) = \left\{ a \in \boldsymbol{A} \mid C_5^-(a) = \max_{b \in \boldsymbol{A}} C_5^-(a) \right\} \tag{37}$$

- The *weak nondomination* of an alternative *a* over all other alternatives representing the degree to which *a* doesn't weakly dominate all the other alternatives is defined as follows:

$$C_6^-(a) = \underset{b \in \boldsymbol{A} \setminus \{a\}}{T_1} [1 - S(a,b)] \tag{38}$$

The choice set corresponding to the weak nondomination (of an alternative on all others) function C_6^- is given by

$$C_6^-(\boldsymbol{A},S) = \left\{ a \in \boldsymbol{A} \mid C_6^-(a) = \max_{b \in \boldsymbol{A}} C_6^-(a) \right\} \tag{39}$$

- The *strict domination* of all alternatives on alternative *a* that gives the degree to which *a* is *strictly dominated* by all the other alternatives is defined as follows:

$$C_7^-(a) = \underset{b \in \boldsymbol{A} \setminus \{a\}}{T_1} P(b,a) \tag{40}$$

The choice set corresponding to the strict domination (of all alternatives on a given alternative) function C_7^- is given by

$$C_7^-(\boldsymbol{A},S) = \left\{ a \in \boldsymbol{A} \mid C_7^-(a) = \max_{b \in \boldsymbol{A}} C_7^-(b) \right\} \tag{41}$$

- The *strict nondomination* of an alternative *a* over all other alternatives representing the degree to which *a doesn't strictly dominate* all the other alternatives is defined as follows:

$$C_8^-(a) = \underset{b \in \boldsymbol{A} \setminus \{a\}}{T_1} [1 - P(a,b)] \tag{42}$$

The choice set corresponding to the strict nondomination (of an alternative on all others) function C_8^- is given by

$$C_8^-(\boldsymbol{A},S) = \left\{ a \in \boldsymbol{A} \mid C_8^-(a) = \max_{b \in \boldsymbol{A}} C_8^-(b) \right\} \tag{43}$$

A ranking method can be obtained using the core concept. Once the set of best alternatives (C_{k+1}) is chosen by the choice function $C(R, \boldsymbol{A}_k)$, which can be any of the choice functions defined above, it is removed from the initial set $\boldsymbol{A}$, and another core set is found between the remaining alternatives ($\boldsymbol{A}_k \setminus C_{k+1}$). This reasoning is applied until the current set ($\boldsymbol{A}_k$) is empty. This algorithm was proposed in (Perny, 1992), and it is described as follows:

Set $k := 0$ and $\boldsymbol{A}_k := \boldsymbol{A}$
  While $\boldsymbol{A}_k \neq \phi$ do
  Begin
    $C_{k+1} := C(R,\boldsymbol{A}_k)$
    $\boldsymbol{A}_{k+1} := \boldsymbol{A}_k \setminus C_{k+1}$
    $k := k+1$
  End

R is one of the relations used to define the first four choice functions (C_1^+ to C_4^+), specifically the weak preference S and strict preference P relations. The resulting preorder $\succeq_R$ is a complete ranking of sets of single or multiple (indifferent) alternatives from best to worst, where R stands for S or P. Four rankings can be obtained using the *strength* concept.

The same algorithm can be used to obtain a second type of preorder but this time using the last four functions (C_5^- to C_8^-). As they are based on the weakness concept, an ascending preorder from worst to the best will be constructed, denoted by $\preceq_R$. These second type of rankings can be different from the previous one.

The notions of ascending–descending and weak–strict rankings are introduced as follows. Similar concepts were used in methods like ELECTRE II and III, MAPPACC, and PRAGMA. Methods like PROMETHEE I and II use concepts of weakness and strength of alternatives but in a different manner. Four different preorders can be defined as follows:

- *Descending weak preorder* is the complete ranking obtained using the iterated choice functions C_1^+ or C_2^+,
- *Descending strict preorder* is the complete ranking obtained using the iterated choice functions C_3^+ or C_4^+,
- *Ascending weak preorder* is the complete ranking obtained using the iterated choice functions C_5^- or C_6^-,
- *Ascending strict preorder* is the complete ranking obtained using the iterated choice functions C_7^- or C_8^-.
- Looking at the choice functions considered, we see that in fact, C_5^-, C_6^-, C_7^-, and C_8^- are "dual" of the functions of C_1^+, C_2^+, C_3^+ and C_4^+ respectively. So each pair $C_1^+ - C_5^-$, $C_2^+ - C_6^-$, $C_3^+ - C_7^-$ and $C_4^+ - C_8^-$ express the force and the weakness, when used in a ranking procedure. At the same time, pairs like $C_1^+ - C_2^+$ and $C_3^+ - C_4^+$ respectively, $C_5^- - C_6^-$ and $C_7^- - C_8^-$ express another type of "duality" that notions of *"outgoing" domination–non domination* (i.e., of an alternative on all the other alternatives), when talking about strength, respectively *"incoming" domination–non domination* (i.e., of all alternatives on the alternative under consideration), when considering the weakness. And

finally, both "dualities" are present for both weak and strict preference relations.

The eight functions can be used alone to obtain a final ranking (weak or strict preorder). Nevertheless, the rankings obtained from two preorders (one descending and the other ascending), thus allowing incomparability (since an alternative a_i may be preferred over another alternative a_j in one preorder and a_j preferred over a_i in the other preorder), are richer and more interesting, as they take into account concepts that may be opposite, or dual, as shown above. Various ranking procedures based on a pair of choice functions, together with their characterization from the following points of view, can be obtained:

- Type of preference: *weak–strict*,
- Type of the ranking of individual choice functions: *ascending–descending*,
- Concept involved: *strength–weakness*,
- Intuitive meaning of the individual choice functions: *incoming domination*, *incoming nondomination*, *outgoing domination*, and *outgoing nondomination*.

4.1.4 Illustrative Example

The example is adapted from (Wang, 2001). Let us consider the seven valve types (a_1 to a_7), and the criteria are cost, maintenance, criteria sensitivity, leakage, rangibility, and stability (g_1 to g_6). The performance matrix is given in Table 5.

Table 5. Performance Matrix for Seven Valve Types (Trapezoidal Fuzzy Numbers)

Alternatives	Criteria's weights of importance					
	0.217	0.174	0.174	0.217	0.087	0.131
	Performance with respect to criterion g_k					
	g_1	g_2	g_3	g_4	g_5	g_6
A_1	(4, 5, 5, 6)	(5, 6, 7, 8)	(7, 8, 8, 9)	(7, 8, 8, 9)	(7, 8, 8, 9)	(1, 2, 2, 3)
A_2	(7, 8, 8, 9)	(8, 9, 10, 10)	(7, 8, 8, 9)	(2, 3, 4, 5)	(8, 9, 10, 10)	(7, 8, 8, 9)
A_3	(7, 8, 8, 9)	(1, 2, 2, 3)	(7, 8, 8, 9)	(7, 8, 8, 9)	(5, 6, 8, 9)	(5, 6, 7, 8)
A_4	(1, 2, 4, 5)	(4, 5, 5, 6)	(4, 5, 5, 6)	(2, 3, 7, 8)	(4, 5, 8, 9)	(8, 9, 10, 10)
A_5	(7, 8, 8, 9)	(5, 6, 7, 8)	(5, 6, 7, 8)	(8, 9, 10, 10)	(1, 2, 2, 3)	(1, 2, 2, 3)
A_6	(4, 5, 5, 6)	(4, 5, 5, 6)	(2, 3, 4, 5)	(5, 6, 7, 8)	(8, 9, 10, 10)	(8, 9, 10, 10)
A_7	(4, 5, 7, 8)	(8, 9, 10, 10)	(7, 8, 8, 9)	(5, 6, 7, 8)	(8, 9, 10, 10)	(7, 8, 8, 9)

The single criterion fuzzy outranking relations S_k are first calculated for each criterion g_k, k = 1…6 using relation (25) for a number of α-cuts N = 50. In the second step, S_k are aggregated using the weighted arithmetic (a particular case of the Choquet intergral) mean with the criteria weights of importance given in Table 5. These steps are repeated for the five representative situations given by the pair of parameters (κ,β), representing the decision maker's attitude. Figures 9−13 represent the above-mentioned situations in terms of the global fuzzy outranking relation S.

	1	2	3	4	5	6	7
1	1	0.413	0.652	0.869	0.62	0.804	0.63
2	0.783	1	0.783	0.902	0.783	0.685	0.783
3	0.773	0.638	1	0.695	0.663	0.618	0.638
4	0.241	0.356	0.388	1	0.306	0.605	0.257
5	0.766	0.461	0.635	0.782	1	0.782	0.461
6	0.512	0.435	0.426	0.853	0.262	1	0.652
7	0.817	0.808	0.624	0.902	0.591	0.902	1

Figure 9. $S(a_i,a_j)$ for $\kappa = 0$, $\beta = \beta^c$ (conserv-pessim)

	1	2	3	4	5	6	7
1	1	0.411	0.652	0.869	0.615	0.802	0.495
2	0.783	1	0.783	0.776	0.783	0.682	0.783
3	0.826	0.66	1	0.695	0.658	0.628	0.66
4	0.354	0.368	0.446	1	0.272	0.77	0.392
5	0.783	0.478	0.652	0.782	1	0.782	0.478
6	0.516	0.435	0.446	0.87	0.245	1	0.519
7	0.837	0.837	0.675	0.899	0.62	0.899	1

Figure 10. $S(a_i,a_j)$ for $\kappa = 1$, $\beta = \beta^c$ (conserv-optim)

	1	2	3	4	5	6	7
1	1	0.404	0.652	0.869	0.598	0.795	0.546
2	0.783	1	0.783	0.822	0.783	0.671	0.783
3	0.796	0.633	1	0.695	0.641	0.617	0.633
4	0.252	0.356	0.402	1	0.261	0.662	0.305
5	0.761	0.456	0.63	0.782	1	0.782	0.456
6	0.484	0.435	0.419	0.848	0.24	1	0.577
7	0.81	0.807	0.617	0.888	0.59	0.888	1

Figure 11. $S(a_i, a_j)$ for $\kappa = 0$, $\beta = \beta^m$ (moderate)

	1	2	3	4	5	6	7
1	1	0.396	0.652	0.869	0.578	0.787	0.613
2	0.783	1	0.783	0.876	0.783	0.659	0.783
3	0.759	0.614	1	0.695	0.621	0.61	0.614
4	0.164	0.35	0.371	1	0.238	0.562	0.214
5	0.745	0.44	0.614	0.782	1	0.782	0.44
6	0.452	0.435	0.399	0.832	0.228	1	0.652
7	0.79	0.788	0.578	0.876	0.571	0.876	1

Figure 12. $S(a_i, a_j)$ for $\kappa = 0$, $\beta = \beta^a$ (agress-pessim)

	1	2	3	4	5	6	7
1	1	0.395	0.652	0.869	0.576	0.786	0.444
2	0.783	1	0.783	0.734	0.783	0.658	0.783
3	0.826	0.62	1	0.695	0.619	0.612	0.62
4	0.248	0.352	0.404	1	0.23	0.712	0.357
5	0.749	0.444	0.618	0.782	1	0.782	0.444
6	0.453	0.435	0.404	0.836	0.224	1	0.484
7	0.795	0.795	0.591	0.875	0.578	0.875	1

Figure 13. $S(a_i, a_j)$ for $\kappa = 1$, $\beta = \beta^a$(agress-optim)

The complete preorders given by the functions $C_1^+ (\equiv C_4^+)$, $C_2^+ (\equiv C_3^+)$, $C_5^- (\equiv C_8^-)$, and $C_6^- (\equiv C_7^-)$ were determined for each of the five representative decision attitudes. Because the fuzzy numbers expressing the performances of the considered alternatives interfere very little and in a trivial manner, we observed an influence that is not strong enough to change the partial preorders when sliding from a conservative to an aggressive attitude. Some changes are noticed when varying the other parameter (κ). Table 6 shows the complete preorders:

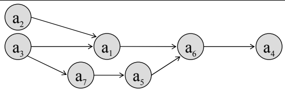
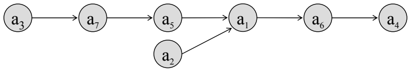
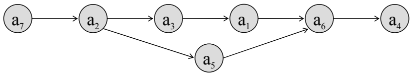
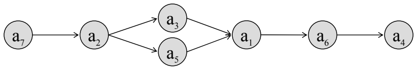
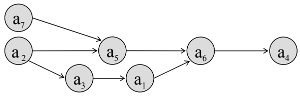
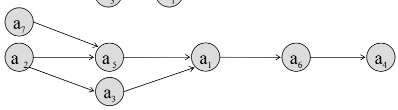
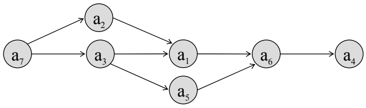
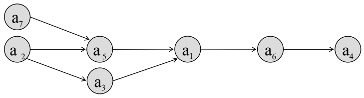
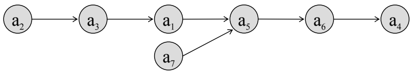
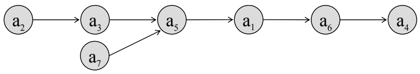
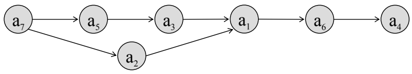
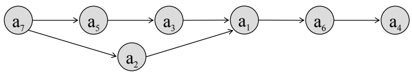
Table 6. Complete Preorder for the Seven Types of Valves

No.	Choice functions	Decision strategy (κ, β)	Preference
1	$C_1^+ \equiv C_4^+$	(0, βa), (0, βc)	2 > 3 > 1 > 7 > 5 > 6 > 4
		(1,βa), (0.5,βm), (1,βc)	2 > 3 > 7 > 5 > 1 > 6 > 4
2	$C_2^+ \equiv C_3^+$	(0, βa), (0, βc)	3, 5, 7 > 2 > 1 > 6 > 4
		(1,βa), (0.5,βm), (1,βc)	3, 5, 7 > 2 > 1 > 6 > 4
3	$C_5^- \equiv C_8^-$	(0, βa), (0, βc)	7 > 2 > 5 > 3 > 1 > 6 > 4
		(1,βa), (0.5,βm), (1,βc)	7 > 2 > 5 > 3 > 1 > 6 > 4
4	$C_6^- \equiv C_7^-$	(0, βa), (0, βc)	7 > 2 > 3 > 1 > 5 > 6 > 4
		(1,βa), (0.5,βm), (1,βc)	7 > 2 > 3 > 5 > 1 > 6 > 4

For each decision strategy, six partial preorders can be derived from the above table by intersecting pairs of choice functions. They are shown in Table 7.

Besides the theoretical foundations of this method, its advantages are related to the practical aspects, namely the format of the input data that can be used (general, nonanalytical representations of fuzzy numbers) where the preference function is described as a vector of α-cuts. Six rankings are proposed. One, several, or all of them can be used to reinforce the choice or the ranking. They enclose different choice ideas, all together offering a large “palette” of concepts. It is up to the decision maker which of them is to be used in the concrete problem. The concepts that are proposed are easy to understand, and they give transparency to the decision process.

Table 7. Partial Ranking of the Eight Valve Types

No.	Choice functions	Decision strategy (κ, β)	Preference
1	$(C_1^+ \equiv C_4^+)$ – $(C_2^+ \equiv C_3^+)$	$(0, \beta^a), (0, \beta^c)$	
		$(1,\beta^a), (0.5,\beta^m), (1,\beta^c)$	
2	$(C_1^+ \equiv C_4^+)$ – $(C_5^- \equiv C_8^-)$	$(0, \beta^a), (0, \beta^c)$	
		$(1,\beta^a), (0.5,\beta^m), (1,\beta^c)$	
3	$(C_5^- \equiv C_8^-)$ – $(C_6^- \equiv C_7^-)$	$(0, \beta^a), (0, \beta^c)$	
		$(1,\beta^a), (0.5,\beta^m), (1,\beta^c)$	
4	$(C_2^+ \equiv C_3^+)$ – $(C_6^- \equiv C_7^-)$	$(0, \beta^a), (0, \beta^c)$	
		$(1,\beta^a), (0.5,\beta^m), (1,\beta^c)$	
5	$(C_1^+ \equiv C_4^+)$ – $(C_6^- \equiv C_7^-)$	$(0, \beta^a), (0, \beta^c)$	
		$(1,\beta^a), (0.5,\beta^m), (1,\beta^c)$	
6	$(C_2^+ \equiv C_3^+)$ – $(C_5^- \equiv C_8^-)$	$(0, \beta^a), (0, \beta^c)$	
		$(1,\beta^a), (0.5,\beta^m), (1,\beta^c)$	

4.2 Other Fuzzy Outranking Methods

All outranking methods briefly described in this subsection consider fuzzy evaluations of alternatives on criteria; therefore, they are fuzzy outranking methods.

4.2.1 Method of Czyżak and Słowiński (1996)

This method is an adaptation of ELECTRE III to the case where the concordance and discordance indices are determined from the fuzzy evaluations of alternatives on criteria through the use of four different measures using possibility and necessity concepts from possibility theory developed in Dubois and Prade (1988). The aggregation of the possibility and necessity measures to drive the concordance and discordance indices is realized through the use of a *weighted root-power mean*. Apart from an adjustment of the monocriterion concordance and discordance indices through some transformation, the rest of the method is similar to ELECTRE III. The method is illustrated through its application to the ground water management problem considered in Duckstein et al. (1994).

4.2.2 Method of Wang (1997)

This method is based on the consideration of a fuzzy preference relation P defined each pair of alternatives (a, b) whose respective fuzzy scores on a given criterion are A and B as follows:

$$P(a,b) = \frac{D(A,B) + D(A \cap B,0)}{D(A,o) + D(B,0)} \tag{44}$$

where $D(A,B)$ represents the areas where A dominates B, $D(A \cap B,0)$ represents the intersection areas of A and B, $D(A,0)$ represents the area of A, and $D(B, 0)$ represents the area of B. It is worth recalling that this fuzzy relation is considered by Tseng and Klein (1989) for the problem of ranking fuzzy numbers.

The outranking relation is defined for:

- The case of a pseudo-order preference model where a preference and indifference thresholds are associated with criteria,
- The case of a semi-order preference model where only indifference thresholds are associated with criteria,

- The case of a complete-preorder preference model where the preference and indifference thresholds are null for each criterion.

The exploitation phase is based on the consideration of the concepts of dominance and non-dominance sets.

The method is illustrated through its application to the problem of evaluating and comparing design concepts in conceptual design.

Güngör and Arikan (2000) applied a similar method to the problem of energy policy planning.

4.2.3 Method of Wang (1999)

In this method, the concordance and discordance indices are determined from the fuzzy evaluations of alternatives on criteria through the use of possibility and necessity measures. More specifically, for two design requirements r_i and r_j, the concordance and discordance indices of criterion C_k with the assertion "r_i is at least as good as r_j" are defined as follows:

$$CI_k(r_i, r_j) = \theta POSS_k(r_i \geq r_j) + (1-\theta) NESS_k(r_i \geq r_j) \quad (45)$$

$$DI_k(r_i, r_j) = NESS_k(r_j \geq r_i) \quad (46)$$

where θ is a preference ratio such that $0 \leq \theta \leq 1$.

The global outranking relation is obtained from monocriterion concordance and discordance indices through the use of the aggregation method developed by Siskos et al. (1984).

The method is illustrated through its application to the problem of prioritizing design requirements in quality function deployment in the case of a car design.

4.2.4 Method of Wang (2001)

In this method, the construction of the fuzzy outranking relation is similar to that of Czyżak and Słowiński (1996) since the concordance and discordance indices are determined from the fuzzy evaluations of alternatives on criteria through the use of four different measures using possibility and necessity concepts. However, the exploitation of the global fuzzy outranking relation is different from ELECTRE III since it is based on the determination of the set of nondominated alternatives as it is considered in Orlovsky (1978). The method is illustrated through its application to the problem of ranking engineering design concepts in conceptual design.

5. CONCLUSION

In this chapter we made a clear distinction between outranking methods based on the construction and exploitation of a valued outranking relation and outranking methods based on the construction and exploitation of a fuzzy outranking relation since they are applicable to two different situations. Indeed, the outranking methods with a valued outranking relation are applicable to the situation where the evaluations of alternatives on criteria are crisp, whereas the outranking methods with a fuzzy outranking relation are applicable to the situation where the evaluations of alternatives on criteria are fuzzy. The outranking methods with a valued outranking relation are called valued outranking methods and the outranking methods with a fuzzy outranking relation are called fuzzy outranking methods. In the literature the fuzzy and valued outranking methods are often confused and the clarification made in this chapter allows avoiding this confusion.

All fuzzy outranking methods deal with the problem of comparing fuzzy numbers; however, they consider different approaches:

- Gheorghe et al. (2004) consider an approach based on α-cuts;
- Czyżak and Słowiński (1996) and Wang (1999, 2001) consider an approach based on possibility and necessity measures;
- Wang (1997) and Güngör and Arikan (2000) consider an approach based on the comparison of areas of fuzzy numbers.

The valued outranking methods ELECTRE III and PROMETHEE are widely applied to real-world problems; however, they are not suitable to the problems where the evaluations of alternatives on criteria are fuzzy. The fuzzy outranking methods presented in this chapter are quite recent compared with the valued outranking methods, and even if they were applied to specific problems, they can be adapted to any MCDA problem where the evaluations of alternatives on criteria are fuzzy.

In this chapter, we provided two illustrative examples, one for a valued outranking method, namely ELECTRE III, and one for a fuzzy outranking method, namely the method developed by the authors. The objective is to show that outranking methods can be applied to various problems with the mention that valued outranking methods are suitable to problems with crisp evaluations (i.e., the case of the treatment of products at their EOL) and fuzzy outranking methods are suitable to problems with fuzzy evaluations (i.e., the case of design concept selection in conceptual design).

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FUZZY MULTI-CRITERIA EVALUATION OF INDUSTRIAL ROBOTIC SYSTEMS USING TOPSIS

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Abstract: Industrial robots have been increasingly used by many manufacturing firms in different industries. Although the number of robot manufacturers is also increasing with many alternative ranges of robots, potential end users are faced with many options in both technical and economical factors in the evaluation of the industrial robotic systems. Industrial robotic system selection is a complex problem, in which many qualitative attributes must be considered. These kinds of attributes make the evaluation process hard and vague. The hierarchical structure is a good approach to describing a complicated system. This chapter proposes a fuzzy hierarchical technique for order preference by similarity ideal solution (TOPSIS) model for the multi-criteria evaluation of the industrial robotic systems. An application is presented with some sensitivity analyses by changing the critical parameters.

Key words: Fuzzy sets, TOPSIS, robotic systems, multi-criteria, hierarchy

1. INTRODUCTION

Robotics is the science and technology of robots, their design, manufacturing and application. Robotics requires a working knowledge of electronics, mechanics, and software, and it is usually accompanied by a large working knowledge of many other subjects. A robot is a mechanical or virtual, artificial agent. It is usually an electromechanical system, which,

C. Kahraman (ed.), Fuzzy Multi-Criteria Decision Making.

by its appearance or movements, conveys a sense that it has intent or agency of its own. The word “robot” can refer to both physical robots and virtual software agents, but the latter are usually referred to as “bots” to differentiate. While there is still discussion about which machines qualify as robots, a typical robot will have several, though not necessarily all of, the following properties (Craig, 2005; Tsai, 1999):

- is not “natural” (i.e., artificially created),
- can sense its environment and manipulate or interact with things in it,
- has some degree of intelligence or ability to make choices based on the environment, or has an automatic control/preprogrammed sequence
- is programmable,
- moves with one or more axes of rotation or translation,
- makes dexterous coordinated movements,
- appears to have intent or agency.

An industrial robot is officially defined by ISO (Anonymous, 2007) as an automatically controlled, reprogrammable, multipurpose manipulator programmable in three or more axes. The field of robotics may be more practically defined as the study, design, and use of robot systems for manufacturing (a top-level definition relying on the prior definition of a robot). During the last decades, both the ranges of applications and the number of available industrial robots have substantially increased. Industrial robots are used for many applications, such as assembly, loading and unloading, material handling, spray painting, and welding. While evaluating industrial robots, potential end users are faced with many options in the selection of an appropriate industrial robot to meet their requirements. The decision in robot selection is therefore more complex because many technical and economical factors affect the performance of the industrial robots. Hence, a multi-criteria evaluation approach is required. Fuzzy sets and systems methodologies are useful for modeling uncertainty and imprecision due to the complexity of contemporary industrial robots, which integrate economical and technical evaluation factors.

Humans are unsuccessful in making quantitative predictions, whereas they are comparatively efficient in qualitative forecasting. Furthermore, humans are more prone to interfere with biasing tendencies if they are forced to provide numerical estimates since the elicitation of numerical estimates forces an individual to operate in a mode that requires more mental effort than that required for less precise verbal statements (Karwowski and Mital, 1986). Since fuzzy linguistic models permit the translation of verbal expressions into numerical ones, thereby dealing quantitatively with imprecision in the expression of the importance of each criterion, some multi-attribute methods based on fuzzy relations can be used. Applications of fuzzy sets within the field of decision making have, for the most part, consisted of extensions or fuzzifications of the classic theories of decision-making. Although decision making under conditions of risk and uncertainty has been modeled by probabilistic decision theories and by game theories, the fuzzy decision theory attempts to deal with the vagueness or fuzziness inherent in subjective or imprecise determinations of preferences, constraints, and goals (Yager, 1982).

Several models have been suggested for the robot selection in the past. They can be classified into five categories: (1) multi-criteria decision making (MCDM) models, (2) production system performance optimization models, (3) computer assisted models, (4) statistical models, and (5) other approaches. MCDM models include multi-attribute decision making (MADM) models (Agrawal et al., 1991; Jones et al., 1985; Nnaji and Yannacopoulou, 1988) multi-objective decision making (MODM) models (Agrawal et al., 1991), and other similar approaches (Huang and Ghandforoush, 1984; Nnaji, 1988). In MADM, all objectives of the decision maker are unified under a superfunction termed the decision maker’s utility, which depends on robot attributes. In MODM, the decision maker’s objective, such as optimal utilization of resources and improved quality, remain explicit and are assigned weights reflecting their relative importance. The main advantage of MCDM models is their ability to consider a large number of robot attributes. Using MCDM, the decision maker can consider engineering, vendor-related, and cost attributes; however, for a problem as complex as robot selection, the data requirements these models place on the decision maker may be overwhelming (Narasimhan and Vickery, 1988). Production system performance optimization models select a robot that optimizes some performance measures of the production system, such as quality or throughput, with robot attributes treated as decision variables. Computer assisted models have been advocated by many researchers to deal with the large number of robot attributes and available robots (Boubekri et al., 1991; Fisher and Maimon, 1988). In general, the decision maker starts by providing the data about the robot application. The data are used by an expert system to provide a list of important robot engineering attributes and their desired values, which in turn is used to obtain a list of feasible robots from a descriptive database of available robots. Statistical models focus on the trade-off between robot engineering attributes and identify robots that provide the best combination of attribute values. Other approaches to the problem include the development of robot time and motion system studies

(Nof, 1985), economic cost/benefit analysis (Nof and Lecthman, 1982) and data envelopment analysis (Knott and Getto, 1982).

In this chapter, fuzzy multi-attribute industrial robotic system selection problem is handled. A fuzzy hierarchical fuzzy Technique for Order Preference by Similarity Ideal Solution (TOPSIS) method is developed to solve this multi-attribute selection problem. In the current literature, the only method that takes the hierarchy among attributes and alternatives into consideration is the analytical hierarchy process (AHP). The developed method, fuzzy hierarchical TOPSIS, also has the ability of considering the hierarchy among attributes and alternatives.

This chapter is organized as follows. In Section 2, the criteria for evaluation of industrial robotic systems are presented. The fuzzy multi-criteria hierarchical TOPSIS method for industrial robot selection is developed in Section 3. Evaluation of industrial robotic systems is illustrated in Section 4. Finally, concluding remarks are given in Section 5.

2. CRITERIA FOR EVALUATION OF INDUSTRIAL ROBOTIC SYSTEMS

Traditional economic analysis techniques incorporate direct costs (and benefits) to which dollar values can be attached. Using these techniques, the evaluation of robots may result in an expected loss or negative return. Management must accurately assess the value of the intangible benefits provided by the investment in automation against the cost figures. If those responsible for the decision and the commitment of company resources do consider the intangible benefits (precision or accuracy, programmability, etc.) to be greater than the cost, then an investment in robots is justified. Thus, evaluation of the industrial robotic systems is often specified using many parameters that can be categorized into two main groups:

1. Economical Attributes
   - Investment Costs (InvC)
     - Purchase Cost (PC): The basic costs of planning and design by the user company's engineering staff to install the robot.
     - Installation Cost (InsC): This includes the labor and materials needed to prepare the installation site.
     - Special Tooling Cost (ST): This includes the cost of the the fixtures and tools required to operate the work cell.

- Operating Costs (OC)
    - o Maintenance Cost (MaC): This covers the anticipated costs of maintenance and repair for the robot cell.
    - o Labor Cost (LC): The direct labor costs associated with the operation of the robot cell and the indirect labor costs that can be directly allocated to the operation of the robot cell.
    - o Training Cost (TC): Training is a continuing activity in which much of it is required for the installation.

2. Technical Attributes
    - Repeatability (Rep): This is a measure of the ability of the robot to return to its original point.
    - Speed (Sp)
    - Memory capacity (MeC)
    - Precision or accuracy (Pre)
    - Programmability (Pro)
    - Number of axes (NA)
    - Workload (Wl)

The hierarchy considered in the study is given in Figure 1.

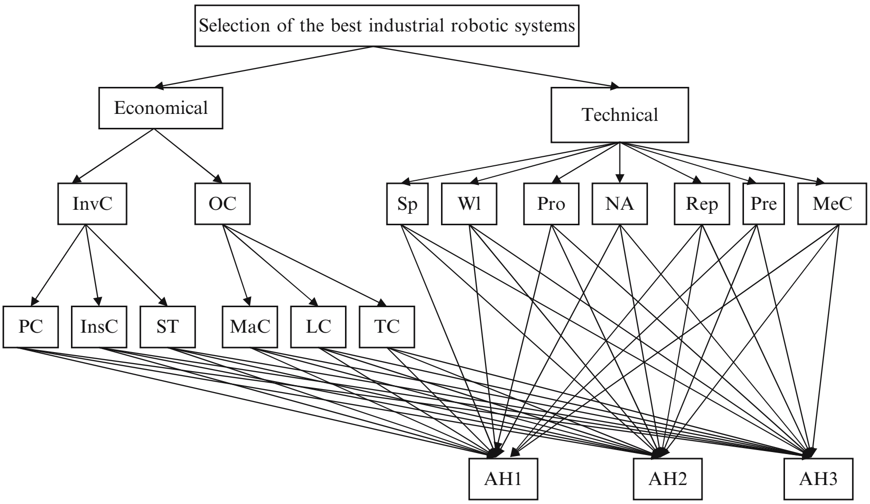

Figure 1. The hierarchy to evaluate industrial robotic systems

3. FUZZY MULTI-ATTRIBUTE DECISION-MAKING METHODS

Fuzzy sets were introduced by Zadeh in 1965 to represent/manipulate data and information possessing nonstatistical uncertainties (Zadeh, 1965). It was specifically designed to represent mathematical uncertainty and vagueness and to provide formalized tools for dealing with the imprecision intrinsic to many problems. Fuzzy logic provides an inference morphology that enables approximate human reasoning capabilities to be applied to knowledge-based systems. The theory of fuzzy logic provides a mathematical strength to capture the uncertainties associated with human cognitive processes, such as thinking and reasoning. Some essential characteristics of fuzzy logic are related to the following: Exact reasoning is viewed as a limiting case of approximate reasoning, everything is a matter of degree, knowledge is interpreted a collection of elastic or, equivalently, fuzzy constraint on a collection of variables, inference is viewed as a process of propagation of elastic constraints, and any logical system can be fuzzified.

Two main characteristics of fuzzy systems give them better performance for specific applications: Fuzzy systems are suitable for approximate reasoning, especially for the system with a mathematical model that is difficult to derive, and fuzzy logic allows decision making with estimated values under incomplete or uncertain information.

Fuzzy multi-criteria decision making (FMCDM) has provoked great interest in decision science, systems engineering, management science, and operations research. Fuzzy multi-attribute decision making is an important component of the FMCDM. Many efficient methods for fuzzy multi-attribute decision making problems exist with the decision maker’s preference information completely known and completely unknown.

The key to solving fuzzy multi-criteria decision making problems is how to obtain preference information of the decision-maker, i.e., criteria weights. Many efficient methods have been presented for the fuzzy multi-criteria decision making problems with the decision maker’s preference information completely known and completely unknown, such as, TOPSIS method, AHP, average weighted comprehensive method, fuzzy optimum seeking method, minimum membership degree method, average weighted programming method, fuzzy neural networks comprehensive decision making method, fuzzy iteration method, and target decision by entropy weight and fuzzy. But, no research exists in fuzzy multi-criteria decision making situated between the above extreme circumstances, i.e., the fuzzy multi-criteria decision making with incomplete information. Therefore, research of such problems is of importance to scientific research and real applications.

3.1 Fuzzy TOPSIS

TOPSIS views a MADM problem with m alternatives as a geometric system with m points in the n-dimensional space. It was developed by Hwang and Yoon (1981). The method is based on the concept that the chosen alternative should have the shortest distance from the positive-ideal solution and the longest distance from the negative-ideal solution. TOPSIS defines an index called similarity (or relative closeness) to the positive-ideal solution and the remoteness from the negative-ideal solution. Then the method chooses an alternative with the maximum similarity to the positive-ideal solution.

Using the vector normalization, the method chooses the alternative with the largest value of C_i^* as given in Eq. 1.

$$C_i^* = \frac{\sqrt{\sum_{j=1}^{n}\left(w_j \frac{x_{ij}}{\sqrt{\sum_{i=1}^{m} x_{ij}^2}} - v_j^-\right)^2}}{\sqrt{\sum_{j=1}^{n}\left(w_j \frac{x_{ij}}{\sqrt{\sum_{i=1}^{m} x_{ij}^2}} - v_j^*\right)^2} + \sqrt{\sum_{j=1}^{n}\left(w_j \frac{x_{ij}}{\sqrt{\sum_{i=1}^{m} x_{ij}^2}} - v_j^-\right)^2}} \tag{1}$$

or it chooses the alternative with the least value of C_i^- formulated as in Eq. 2.

$$C_i^- = \frac{\sqrt{\sum_{j=1}^{n}\left(w_j \frac{x_{ij}}{\sqrt{\sum_{i=1}^{m} x_{ij}^2}} - v_j^*\right)^2}}{\sqrt{\sum_{j=1}^{n}\left(w_j \frac{x_{ij}}{\sqrt{\sum_{i=1}^{m} x_{ij}^2}} - v_j^*\right)^2} + \sqrt{\sum_{j=1}^{n}\left(w_j \frac{x_{ij}}{\sqrt{\sum_{i=1}^{m} x_{ij}^2}} - v_j^-\right)^2}} \tag{2}$$

where i (i = 1 ,…, m) and j (j = 1 ,…, n) are index numbers for the alternatives and attributes, respectively; w_j is the weight of the jth attribute; x_{ij} is the attribute rating for ith alternative's jth attribute; v_j^* is the positive-ideal value for jth attribute, where it is a maximum for benefit attributes and a minimum for cost attributes; and v_j^- is the negative-ideal value for the jth attribute, where it is a minimum for benefit attributes and a maximum for cost attributes.

In the last decade, some fuzzy TOPSIS methods were developed in the literature: Chen and Hwang (1992) transform Hwang and Yoon's (1981) method into a fuzzy case. Liang (1991) presents a fuzzy multi-criteria decision making based on the concepts of ideal and anti-ideal points. The concepts of fuzzy set theory and hierarchical structure analysis are used to develop a weighted suitability decision matrix to evaluate the weighted suitability of different alternatives versus criteria. Triantaphyllou and Lin (1996) develop a fuzzy version of the TOPSIS method based on fuzzy arithmetic operations, which leads to a fuzzy relative closeness for each alternative. This fuzzy TOPSIS method offers a fuzzy relative closeness for each alternative; the closeness is badly distorted and over-exaggerated because of the reason of fuzzy arithmetic operations. Chen (2000) describes the rating of each alternative and the weight of each criterion by linguistic terms, which can be expressed in triangular fuzzy numbers. Then, a vertex method for TOPSIS is proposed to calculate the distance between two triangular fuzzy numbers. Cheng et al. (2002) apply Chen and Hwang's (1992) fuzzy TOPSIS approach for solving the solid waste management problem in Regina of Saskatchewan Canada. Additionally, they apply four other MCDM methods for the analysis of solid-waste management systems, including simple weighted addition (SWA) method, weighted product (WP) method, cooperative game theory, and ELECTRE. Since all methods result in different rankings of the alternative solutions, they use an aggregation approach called the average ranking procedure to analyze the results. Zhang and Lu (2003) present an integrated fuzzy group decision making method in order to deal with the fuzziness of preferences of the decision-makers.

In this chapter, the weights of the criteria are crisp values gathered by pair-wise comparisons where the preferences of the decision makers are represented by triangular fuzzy numbers (TFNs). Chen and Tzeng (2004) transform a fuzzy MCDM problem into a nonfuzzy MCDM using a fuzzy integral. Instead of using distance, they employ a gray relation grade to define the relative closeness of each alternative. Abo-Sinna and Abou-El-Enien (2005) extend the technique for order preference by similarity ideal solution (TOPSIS) for solving large-scale multiple objective programming problems involving fuzzy parameters. Wang and Elhag (2006) present a nonlinear programming (NLP) solution procedure using a fuzzy TOPSIS method based on alpha level set. They discuss the relationship between the fuzzy TOPSIS method and the fuzzy weighted average (FWA). They illustrate three examples about bridge risk assessments to compare the proposed fuzzy TOPSIS and other procedure. Jahanshahloo et al. (2006) study the cases in which determining precisely the exact value of the attributes is difficult, and as a result of this, the attribute values should be considered as intervals. They aim to extend the TOPSIS method for decision making problems with interval data. By extension of the TOPSIS method, they present an algorithm to determine the most preferable choice among all possible choices, when data are interval.

Table 1. A Comparison of Fuzzy TOPSIS Methods

Source	Attribute Weights	Type of Fuzzy Numbers	Ranking Method	Normalization Method
Chen and Hwang (1992)	Fuzzy Numbers	Trapezoidal	Lee and Li's (1988) generalized mean method	Linear Normalization
Liang (1999)	Fuzzy Numbers	Trapezoidal	Chen's (1985) ranking with maximizing set and minimizing set	Manhattan distance
Chen (2000)	Fuzzy Numbers	Triangular	Chen (2000) proposes vertex method	Linear Normalization
Chu (2002)	Fuzzy Numbers	Triangular	Liou and Wang's (1992) ranking method of total integral value with α=1/2	Modified Manhattan distance
Tsaur et al. (2002)	Crisp Values	Triangular	Zhao and Govind's (1991) center of area method	Vector Normalization
Zhang and Lu (2003)	Crisp Values	Triangular	Chen's (2000) vertex method	Manhattan distance
Chu and Lin (2003)	Fuzzy Numbers	Triangular	Kaufmann and Gupta's (1988) mean of the removals method	Linear Normalization
Cha and Yung (2003)	Crisp Values	Triangular	Cha and Yung (2003) propose a fuzzy distance operator	Linear Normalization
Yang and Hung (2005)	Fuzzy Numbers	Triangular	Chen's (2000) vertex method	Normalized fuzzy linguistic ratings are used
Wang and Elhag (2006)	Fuzzy Number	Triangular	Chen's (2000) vertex method	Linear Normalization
Jahanshahloo et al. (2006)	Crisp Values	Interval data	Jahanshahloo et al. (2006) propose a new normalization & ranking method	

A comparison of the fuzzy TOPSIS methods in the literature is given in Table 1. The comparison includes the computational differences among the methods. In this chapter, we prefer Chen and Hwang's (1992) fuzzy TOPSIS method since the other fuzzy TOPSIS methods are derived from this method with minor differences.

In the following discussion, the steps of fuzzy TOPSIS developed by Chen and Hwang (1992) are given. First, a decision matrix, D, of $m \times n$ dimension is defined as in Eq. 3.

$$D = \begin{array}{c} \\ A_1 \\ \vdots \\ A_i \\ \vdots \\ A_m \end{array} \begin{array}{c} \begin{array}{ccccc} X_1 & \cdots & X_j & \cdots & X_n \end{array} \\ \left[\begin{array}{ccccc} x_{11} & \cdots & x_{1j} & \cdots & x_{1n} \\ \vdots & & \vdots & & \vdots \\ x_{i1} & \cdots & x_{ij} & \cdots & x_{in} \\ \vdots & & \vdots & & \vdots \\ x_{m1} & \cdots & x_{mj} & \cdots & x_{mn} \end{array}\right] \end{array} \qquad (3)$$

where $x_{ij}, \forall i, j$ may be crisp or fuzzy. If x_{ij} is fuzzy, it is represented by a trapezoidal number as $x_{ij} = (a_{ij}, b_{ij}, c_{ij}, d_{ij})$ shown in Figure 2. The fuzzy weights can be described by Eq. 4.

$$w_j = (\alpha_{ij}, \beta_{ij}, \chi_{ij}, \delta_{ij}) \qquad (4)$$

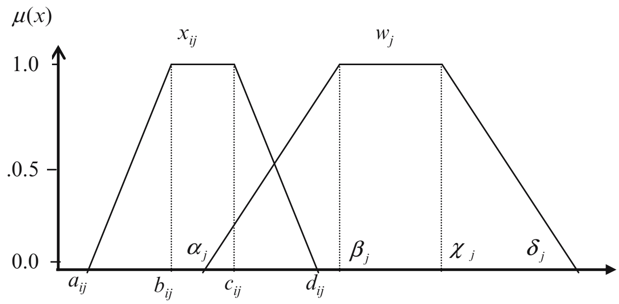

Figure 2. Trapezoidal fuzzy numbers

3.2 Algorithm

The problem is solved using the following steps.

Step 1. Normalize the Decision Matrix. The decision matrix must first be normalized so that the elements are unit-free. To avoid the complicated normalization formula used in classic TOPSIS, we use linear scale transformation as follows:

$$r_{ij} = \begin{cases} x_{ij} / x_j^*, \forall j, x_j \text{ is a benefit attribute} \\ x_j^- / x_{ij}, \forall j, x_j \text{ is a cost attribute} \end{cases} \tag{5}$$

By applying Eq. 5, we can rewrite the decision matrix in Eq. 3 as in Eq. 6.

$$D' = \begin{matrix} \\ A_1 \\ \vdots \\ A_i \\ \vdots \\ A_m \end{matrix} \begin{matrix} X_1 & \cdots & X_j & \cdots & X_n \\ \begin{bmatrix} r_{11} & \cdots & r_{1j} & \cdots & r_{1n} \\ \vdots & & \vdots & & \vdots \\ r_{i1} & \cdots & r_{ij} & \cdots & r_{in} \\ \vdots & & \vdots & & \vdots \\ r_{m1} & \cdots & r_{mj} & \cdots & r_{mn} \end{bmatrix} \end{matrix} \tag{6}$$

When x_{ij} is crisp, its corresponding r_{ij} must be crisp; when x_{ij} is fuzzy, its corresponding r_{ij} must be fuzzy. Eq. 5 is then replaced by the following fuzzy operations: Let $x_{ij} = (a_{ij}, b_{ij}, c_{ij}, d_{ij})$ and $x_j^* = (a_j^*, b_j^*, c_j^*, d_j^*)$, we have:

$$r_{ij} = \begin{cases} x_{ij}(+)x_j^* = \left(\dfrac{a_{ij}}{d_j^*}, \dfrac{b_{ij}}{c_j^*}, \dfrac{c_{ij}}{b_j^*}, \dfrac{d_{ij}}{a_j^*} \right) \\ x_j^-(+)x_{ij} = \left(\dfrac{a_i^-}{d_{ij}}, \dfrac{b_i^-}{c_{ij}}, \dfrac{c_i^-}{b_{ij}}, \dfrac{d_i^-}{a_{ij}} \right) \end{cases} \tag{7}$$

Step 2. Obtain the Weighted Normalized Decision Matrix. This matrix is obtained using

$$v_{ij} = r_{ij} w_j, \forall j, j \tag{8}$$

When both r_{ij} and w_{ij} are crisp, v_{ij} is crisp. When either r_{ij} or w_{ij} (or both) are fuzzy, Eq. 8 may be replaced by the following fuzzy operations:

$$v_{ij} = r_{ij}(.)w_j^* = \left(\frac{a_{ij}}{d_j^*}\alpha_j, \frac{b_{ij}}{c_j^*}\beta_j, \frac{c_{ij}}{b_j^*}\chi_j, \frac{d_{ij}}{a_j^*}\delta_j \right) \tag{9}$$

$$v_{ij} = r_{ij}(.)w_j^* = \left(\frac{a_i^-}{d_{ij}}\alpha_j, \frac{b_i^-}{c_{ij}}\beta_j, \frac{c_i^-}{b_{ij}}\chi_j, \frac{d_i^-}{a_{ij}}\delta_j \right) \tag{10}$$

Eq. 9 is used when the jth attribute is a benefit attribute. Eq. 10 is used when the jth attribute is a cost attribute. The result of Eqs. 9 and 10 can be summarized as in Eq. 11.

$$v = \begin{array}{c} \\ A_1 \\ \vdots \\ A_i \\ \vdots \\ A_m \end{array} \begin{array}{c} \begin{array}{ccccc} X_1 & \cdots & X_j & \cdots & X_n \end{array} \\ \begin{bmatrix} v_{11} & \cdots & v_{1j} & \cdots & v_{1n} \\ \vdots & & \vdots & & \vdots \\ v_{i1} & \cdots & v_{ij} & \cdots & v_{in} \\ \vdots & & \vdots & & \vdots \\ v_{m1} & \cdots & v_{mj} & \cdots & v_{mn} \end{bmatrix} \end{array} \tag{11}$$

Step 3. Obtain the Positive Ideal Solution (PIS), A^*, and the Negative Ideal Solution (NIS), $A-$. PIS and NIS are defined as

$$A^* = \left[v_1^*, \ldots, v_n^*\right] \tag{12}$$

$$A^- = \left[v_1^-, \ldots, v_n^-\right] \tag{13}$$

where $v_j^* = \max_i v_{ij}$ and $v_j^- = \min_i v_{ij}$.

For crisp data, v_j^* and v_j^- are obtained in a straight forward manner. In the case of fuzzy data, v_j^* and v_j^- may be obtained through some ranking procedures. Chen and Hwang use Lee and Li's ranking method for comparison of fuzzy numbers. The v_j^* and v_j^- are the fuzzy numbers with the largest generalized mean and the smallest generalized mean, respectively. The generalized mean for fuzzy number $v_{ij}, \forall i, j,$ is defined as

$$M(v_{ij}) = \frac{-a_{ij}^2 - b_{ij}^2 + c_{ij}^2 + d_{ij}^2 - a_{ij}b_{ij} + c_{ij}d_{ij}}{\left[3\left(-a_{ij} - b_{ij} + c_{ij} + d_{ij}\right)\right]} \tag{14}$$

For each column j, we find a v_{ij} whose greatest mean is v_j^* and whose lowest mean is v_j^-.

Step 4. Obtain the Separation Measures S_i^* and S_i-. In the classic case, seperation measures are defined as:

$$S_i^* = \sum_{j=1}^{n} D_{ij}^*, \quad i = 1,...,m \tag{15}$$

and,

$$S_i^- = \sum_{j=1}^{n} D_{ij}^-, \quad i = 1,...,m. \tag{16}$$

For crisp data, the difference measures D_{ij}^* and D_{ij}^- are given as

$$D_{ij}^* = \left| v_{ij} - v_j^* \right| \tag{17}$$

$$D_{ij}^- = \left| v_{ij} - v_j^- \right|. \tag{18}$$

The computation is straightforward. For fuzzy data, the difference between two fuzzy numbers $\mu_{v_{ij}}(x)$ and $\mu_{v_j^*}(x)$ (based on Zadeh (1965)'s study) is explained as given in Eq. 19.

$$D_{ij}^* = 1 - \sup_x \left[\mu_{v_{ij}}(x) \wedge \mu_{v_j^*}(x) \right] = 1 - L_{ij}, \forall i, j \tag{19}$$

where L_{ij} is the highest degree of similarity of v_{ij} and v_j^*. The value of L_{ij} is best depicted in Figure 3.

Similarly, the difference between $\mu_{v_{ij}}(x)$ and $\mu_{v_j^-}(x)$ is defined as

$$D_{ij}^- = 1 - \sup_x \left[\mu_{v_{ij}}(x) \wedge \mu_{v_j^-}(x) \right] = 1 - L_{ij}, \forall i, j \quad . \tag{20}$$

Note that both D_{ij}^*, D_{ij}^- are crisp numbers.

Step 5. Compute the Relative Closeness to Ideals. This index is used to combine S_i^* and S_i^- indices calculated in Step 4. Since S_i^* and S_i^- are crisp numbers, they can be combined:

$$C_i = S_i^- / \left(S_i^* + S_i^-\right) \tag{21}$$

The alternatives are ranked in descending order of the C_i index.

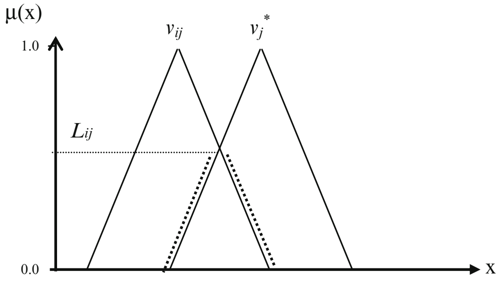

Figure 3. The derivation of L_{ij}

3.3 Fuzzy Hierarchical TOPSIS

In the literature, one of the most known and widely used multi-attribute decision making methods is fuzzy AHP. There are two main differences between AHP and TOPSIS. (1) Pair-wise comparisons for attributes and alternatives are made in AHP, although there is no pair-wise comparison in TOPSIS. (2) AHP uses a hierarchy of attributes and alternatives, whereas TOPSIS does not. The consideration of the hierarchies in the multi-attribute problems provides a great superiority to AHP. The developed fuzzy TOPSIS methods today do not take the hierarchies in the multi-attribute problems into consideration. In the following discussion, we develop a fuzzy hierarchical TOPSIS to solve multi-attribute hierarchical problems. The fuzzy TOPSIS algorithm considering a hierarchy is developed below. The hierarchy given in Figure 4 will be considered.

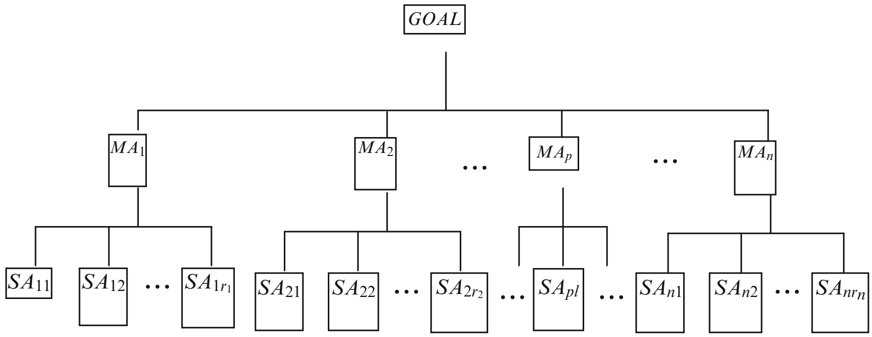

Figure 4. The hierarchy considered in fuzzy TOPSIS algorithm

Assume that we have n main attributes, m sub-attributes, k alternatives, and s respondents. Each main attribute has r_i sub-attributes where the total number of sub-attributes m is equal to the sum of r_i, i = 1,2,3,…n.

The first matrix ($\tilde{I}_{MA}$), given by Eq. 22, is constructed from the weights of the main attributes with respect to the goal.

$$\tilde{I}_{MA} = \begin{array}{c} \\ MA_1 \\ MA_2 \\ \vdots \\ MA_p \\ \vdots \\ MA_n \end{array} \begin{array}{c} \text{Goal} \\ \begin{bmatrix} \tilde{w}_1 \\ \tilde{w}_2 \\ \vdots \\ \tilde{w}_p \\ \vdots \\ \tilde{w}_n \end{bmatrix} \end{array} \tag{22}$$

where $\tilde{w}_p$ is the arithmetic mean of the weights assigned by the respondents and is calculated by Eq. 23

$$\tilde{w}_p = \frac{\sum_{i=1}^{s} \tilde{w}_{pi}}{s}, \quad p = 1,2,...,n \tag{23}$$

where $\tilde{w}_{pi}$ denotes the fuzzy evaluation score of the pth main attribute with respect to the goal assessed by the ith respondent. The second matrix ($\tilde{I}_{SA}$) represents the weights of the sub-attributes with respect to the main attributes. The weights vector obtained from $\tilde{I}_{MA}$ is written above this $\tilde{I}_{SA}$ as illustrated in Eq. 24.

$$\begin{array}{cc} & \begin{array}{cccccc} \tilde{w}_1 & \tilde{w}_2 & \cdots & \tilde{w}_p & \cdots & \tilde{w}_n \\ MA_1 & MA_2 & \cdots & MA_p & \cdots & MA_n \end{array} \\ \tilde{I}_{SA} = \begin{array}{c} SA_{11} \\ SA_{12} \\ \vdots \\ SA_{1r_1} \\ SA_{21} \\ SA_{22} \\ \vdots \\ SA_{2r_2} \\ \vdots \\ SA_{pl} \\ \vdots \\ SA_{n1} \\ SA_{n2} \\ \vdots \\ SA_{nr_n} \end{array} & \left[\begin{array}{cccccc} \tilde{w}_{11} & 0 & \cdots & 0 & \cdots & 0 \\ \tilde{w}_{12} & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots & & \vdots \\ \tilde{w}_{1r_1} & 0 & \cdots & 0 & \cdots & 0 \\ 0 & \tilde{w}_{21} & \cdots & 0 & \cdots & 0 \\ 0 & \tilde{w}_{22} & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots & & \vdots \\ 0 & \tilde{w}_{2r_2} & \cdots & 0 & \cdots & 0 \\ \vdots & 0 & & & & \vdots \\ 0 & 0 & \cdots & \tilde{w}_{pl} & \cdots & 0 \\ \vdots & \vdots & & & & 0 \\ 0 & 0 & \cdots & 0 & \cdots & \tilde{w}_{n1} \\ 0 & 0 & \cdots & 0 & \cdots & \tilde{w}_{n2} \\ \vdots & \vdots & & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & \tilde{w}_{nr_n} \end{array} \right] \end{array} \quad (24)$$

where $\tilde{w}_{pl}$ is the arithmetic mean of the weights assigned by the respondents and is calculated by Eq. 25.

$$\tilde{w}_{pl} = \frac{\sum_{i=1}^{s} \tilde{w}_{pli}}{s} \quad (25)$$

where $\tilde{w}_{pli}$ is the weight of the lth sub-attribute with respect to the pth main attribute assessed by the ith respondent. The third matrix ($\tilde{I}_A$) is formed by the scores of the alternatives with respect to the sub-attributes. The weights vector obtained from $\tilde{I}_{SA}$ are written above this $\tilde{I}_A$ as in Eq. 26.

$$\tilde{I}_A = \begin{array}{c} \\ \\ A_1 \\ A_2 \\ \vdots \\ A_q \\ \vdots \\ A_k \end{array} \begin{array}{c} \tilde{W}_{11} \quad \tilde{W}_{12} \quad \ldots \quad \tilde{W}_{1r_1} \quad \ldots \quad \tilde{W}_{pl} \quad \ldots \quad \tilde{W}_{nr_n} \\ SA_{11} \quad SA_{12} \quad \ldots \quad SA_{1r_1} \quad \ldots \quad SA_{pl} \quad \ldots \quad SA_{nr_n} \\ \begin{bmatrix} \tilde{c}_{111} & \tilde{c}_{112} & \ldots & \tilde{c}_{11r_1} & \ldots & \tilde{c}_{1pl} & \ldots & \tilde{c}_{1nr_n} \\ \tilde{c}_{211} & \tilde{c}_{212} & \ldots & \tilde{c}_{21r_1} & \ldots & \tilde{c}_{2pl} & \ldots & \tilde{c}_{2nr_n} \\ \vdots & \vdots & & \vdots & & \vdots & & \vdots \\ \tilde{c}_{q11} & \tilde{c}_{q12} & \ldots & \tilde{c}_{q1r_1} & \ldots & \tilde{c}_{qpl} & \ldots & \tilde{c}_{qnr_n} \\ \vdots & \vdots & & \vdots & & \vdots & & \vdots \\ \tilde{c}_{k11} & \tilde{c}_{k12} & \ldots & \tilde{c}_{k1r_1} & \ldots & \tilde{c}_{kpl} & \ldots & \tilde{c}_{knr_n} \end{bmatrix} \end{array} \tag{26}$$

where

$$\tilde{W}_{pl} = \sum_{j=1}^{n} \tilde{w}_p \tilde{w}_{pj} \,. \tag{27}$$

Since $w_{pj} = 0$ for $j \neq l$, we can use Eq. 28 instead of Eq. 27

$$\tilde{W}_{pl} = \tilde{w}_p \tilde{w}_{pl} \,. \tag{28}$$

In $\tilde{I}_A$, $\tilde{c}_{qpl}$ is the arithmetic mean of the scores assigned by the respondents, and it is calculated by Eq. 29

$$\tilde{c}_{qpl} = \frac{\sum_{i=1}^{s} \tilde{c}_{qpli}}{s} \tag{29}$$

where $\tilde{c}_{qpli}$ is the fuzzy evaluation score of the qth alternative with respect to the lth sub-attribute under the pth main attribute assessed by the ith respondent. To determine the importance degree of each main attribute with respect to the goal and each sub-attribute with respect to the main-attributes, Table 2 is used. The linguistic terms represented by TFNs for scoring the alternatives under the sub-attributes are given in Table 3.

Table 2. The Importance Degrees

Very low	(0, 0, 0.2)
Low	(0, 0.2, 0.4)
Medium	(0.3, 0.5, 0.7)
High	(0.6, 0.8, 1)
Very High	(0.8, 1, 1)

Table 3. The Scores

Very low	(0, 0, 20)
Low	(0, 20, 40)
Medium	(30, 50, 70)
High	(60, 80, 100)
Very High	(80, 100, 100)

4. EVALUATION OF INDUSTRIAL ROBOTIC SYSTEMS

Three different industrial robotic systems are evaluated using the multi-attribute decision making technique given above. Taking the hierarchy given in Figure 1 into consideration, a questionnaire for fuzzy TOPSIS was prepared to receive the weights of main, sub, and sub-sub-attributes from the experts. A part of this questionnaire is given in Appendix A. The questionnaire is applied to a big automotive company in Turkey. Twenty-four professionals in a company where 4 of them are top managers, 8 of them are division managers of related departments, and 12 of them are engineers, were interviewed. The response rate was 100% with a high support of top management.

First, equations in the fuzzy TOPSIS algorithm using trapezoidal fuzzy numbers given in Section 3.1 are rewritten for TFNs that are considered in this application. Since a TFN (*a, b, c*) can be represented in trapezoidal form as (*a, b, b, c*), it can be easily seen that Eq. 7 can be expressed as follows:

$$r_{ij} = \begin{cases} x_{ij}(+)x_j^* = \left(\dfrac{a_{ij}}{d_j^*}, \dfrac{b_{ij}}{b_j^*}, \dfrac{d_{ij}}{a_j^*}\right) \\ x_j^-(+)x_{ij} = \left(\dfrac{a_i^-}{d_{ij}}, \dfrac{b_i^-}{b_{ij}}, \dfrac{d_i^-}{a_{ij}}\right) \end{cases} \tag{30}$$

Then, Eq. 14 is reduced to Eq. 31.

$$M(v_{ij}) = \frac{-a_{ij}^2 + d_{ij}^2 - a_{ij}b_{ij} + b_{ij}d_{ij}}{\left[3\left(-a_{ij} + d_{ij}\right)\right]} \tag{31}$$

D_{ij}^* and D_{ij}^- are calculated by the Eqs. 32 and 33, respectively.

$$D_{ij}^* = \begin{cases} 1 - \dfrac{c_{ij} - a^*}{b^* + c_{ij} - a^* - b_{ij}} & \text{for } b_{ij} < b^* \\ 1 - \dfrac{c^* - a_{ij}}{b_{ij} + c^* - a_{ij} - b^*} & \text{for } b^* < b_{ij} \end{cases} \quad \forall i, j \tag{32}$$

$$D_{ij}^- = \begin{cases} 1 - \dfrac{c^- - a_{ij}}{b_{ij} + c^- - a_{ij} - b^-} & \text{for } b^- < b_{ij} \\ 1 - \dfrac{c_{ij} - a^-}{b^- + c_{ij} - a^- - b_{ij}} & \text{for } b_{ij} < b^- \end{cases} \quad \forall i, j \tag{33}$$

where $v_j^* = (a^*, b^*, c^*)$ and $v_j^- = (a^-, b^-, c^-)$ are the fuzzy numbers with the largest generalized mean and the smallest generalized mean, respectively.

Then, the steps of the hierarchical fuzzy TOPSIS algorithm are executed. Our model has two main attributes, nine sub-attributes, six sub-sub-attributes, and three alternatives. Evaluations from all 24 respondents are taken, and $\tilde{I}_{MA}$, $\tilde{I}_{SA}$, and $\tilde{I}_{A}$ are obtained and given in Tables 4–7.

Table 4. $\tilde{I}_{MA}$

	GOAL
EA	(0.54, 0.83, 0.91)
TA	(0.32, 0.71, 0.86)

Table 5. $\tilde{I}_{SA}$

	EA	**TA**
InvC	(0.27, 0.69, 0.87)	0
OC	(0.32, 0.61, 0.83)	0
SP	0	(0.54, 0.76, 0.81)
Wl	0	(0.63, 0.78, 0.86)
Pro	0	(0.45, 0.53, 0.74)
NA	0	(0.16, 0.41, 0.56)
Rep	0	(0.21, 0.56, 0.71)
Pre	0	(0.35, 0.78, 0.86)
MeC	0	(0.41, 0.68, 0.87)

Table 6. $\widetilde{I}_{SSA}$

	Inv	OC
PC	(0.53, 0.78, 0.87)	0
InsC	(0.45, 0.53, 0.74)	0
ST	(0.35, 0.42, 0.69)	0
Mac	0	(0.63, 0.78, 0.86)
LC	0	(0.61, 0.88, 0.93)
TC	0	(0.21, 0.56, 0.71)

Table 7. $\widetilde{I}_{A}$

	AH-1	AH-2	AH-3
PC	(17, 39, 56)	(45, 78, 89)	(21, 28, 46)
InsC	(45, 78, 89)	(32, 45, 76)	(13, 26, 57)
STF	(23, 38, 48)	(38, 44, 49)	(41, 53, 62)
Mac	(10, 21, 36)	(29, 39, 52)	(41, 53, 62)
LC	(34, 51, 63)	(25, 33, 37)	(9, 17, 33)
TC	(13, 42, 67)	(32, 45, 76)	(6, 22, 46)
SP	(24, 54, 81)	(13, 34, 56)	(29, 36, 48)
Wl	(16, 31, 46)	(23, 35, 56)	(32, 45, 76)
Pro	(46, 69, 81)	(47, 61, 76)	(29, 36, 48)
NA	(15, 32, 43)	(9, 21, 39)	(39, 63, 81)
Rep	(32, 44, 53)	(27, 34, 55)	(15, 32, 43)
Pre	(13, 45, 72)	(21, 56, 78)	(36, 67, 85)
MeC	(56, 65, 78)	(17, 32, 29)	(44, 72, 87)
Wl	(16, 31, 46)	(23, 35, 56)	(32, 45, 76)

The tables to obtain r_{ij}, v_{ij}, $M(v_{ij})$, D^{*}_{ij}, and D^{-}_{ij} are given in Appendix B. Table 8 shows the distances from the ideal solution for each AH and the normalized values which makes the results' interpretation easier.

Table 8. Distances from the Ideal Solution

	S_i*	S_i-	C_i	Normalized C_i
AH-1	1.184733	1.791474	0.601932	0.39
AH-2	1.644703	1.320108	0.445259	0.29
AH-3	1.508918	1.440309	0.488368	0.32

The results in Table 8 indicate that AH-1 achieves the highest performance, whereas AH-2 has the lowest.

To analyze the attitude of the alternatives under different main attribute weights, a sensitivity analysis is made. The results of sensitivity analyses are given in Table 9 and Figure 5. In Table 9, the states where one of the main attributes has the maximum weight, whereas the other that has less values given in Table 2 are examined. For each state, Normalized Relative Closeness to Ideals (C_i) is computed. Figures 5 and 6 illustrate the graphical representation of these results.

Table 9. The Results of Sensitivity Analyses

			Normalized Ci		
EA	**TA**	**States**	**AH-1**	**AH-2**	**AH-3**
0.8, 1, 1	0.8, 1, 1	1	0.39	0.28	0.32
0.8, 1, 1	0.6, 0.8, 1	2	0.39	0.29	0.31
0.8, 1, 1	0.3, 0.5, 0.7	3	0.38	0.30	0.30
0.8, 1, 1	0, 0.2, 0.4	4	0.38	0.32	0.29
0.8, 1, 1	0, 0, 0.2	5	0.35	0.44	0.19
EA	**TA**	**States**	**AH-1**	**AH-2**	**AH-3**
0.8, 1, 1	0.8, 1, 1	1	0.39	0.28	0.32
0.6, 0.8, 1	0.8, 1, 1	2	0.39	0.26	0.33
0.3, 0.5, 0.7	0.8, 1, 1	3	0.40	0.26	0.33
0, 0.2, 0.4	0.8, 1, 1	4	0.40	0.23	0.35
0, 0, 0.2	0.8, 1, 1	5	0.42	0.14	0.42

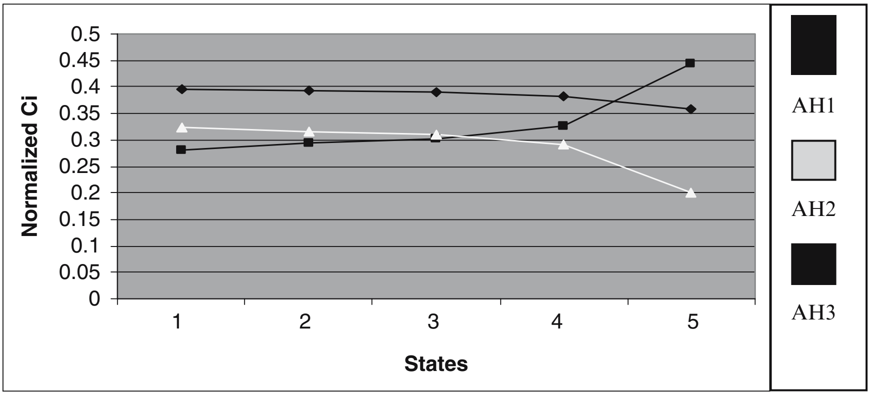

Figure 5. Sensitivity analysis for the case where EA has the highest weight

As shown in Figure 5, the importance of technological attributes decreases, whereas the score of AH-2 increases dramatically and the score of AH-1 decreases. This means AH-2 has superior economical properties. Figure 6 shows that AH-1 gets more superior to the others as importance of economical attributes decreases. This is because AH-1 has better scores than the others on the technical attributes.

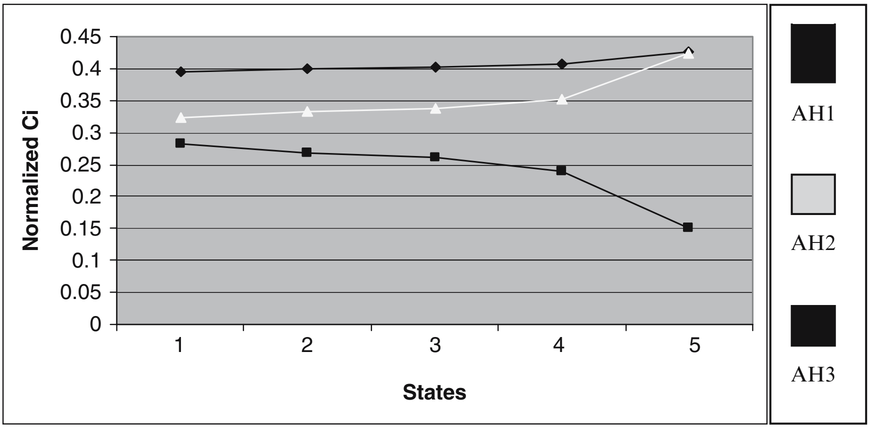

Figure 6. Sensitivity analysis for the case where TA has the highest weight

5. CONCLUSION

In this chapter, a model for evaluating and selecting among industrial robotic systems has been presented. The model is based on the premise that industrial robotic systems selection should be viewed as a product of economical and technical attributes. Economical attributes consist of investment costs and operating costs, whereas technical attributes consist of memory capacity, speed, and number of axes, precision, programmability, repeatability, and workload. In addition, purchasing costs, installation costs, and special tooling are the sub-attributes of investment costs and maintenance. Labor and training costs are the sub-attributes of operating costs. Industrial robotic system selection is a complex problem in which many qualitative attributes must be considered. These kinds of attributes make the evaluation process hard and vague. The hierarchical structure is a good approach to describe a complicated system. The judgments from experts are always vague rather than crisp. It is suitable and flexible to express the judgments of experts in fuzzy numbers instead of in crisp numbers. Fuzzy AHP has the capability of taking pair-wise comparisons of these attributes into account with a hierarchical structure. Many fuzzy TOPSIS methods have been proposed without considering these pair-wise comparisons between attributes and a hierarchical structure by today. To be able to benefit from the superiority of a hierarchical structure, a hierarchical fuzzy TOPSIS method has been developed. It is clear that the selection of an industrial robotic system is a difficult and sensitive issue that has quantitative and qualitative aspects, complexity, and imprecision. However, the developed fuzzy method seems to be usable for the solution of this problem. For additional research, a hierarchical fuzzy TOPSIS method that can take pair-wise comparisons between main and sub-attributes into account in a different manner from AHP may be developed.

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APPENDIX A: QUESTIONNAIRE FOR FUZZY TOPSIS

With respect to the overall goal "Selection of the best industrial robotic systems"

Q1. What degree of importance do you assign to the main attribute Economic Attributes?

Q2. What degree of importance do you assign to the main attribute Technical Attributes?

With respect to: Overall goal		Importance of one attribute with respect to overall goal				
Questions	Attributes	(0, 0, 0.2) Very Low	(0, 0.2, 0.4) Low	(0.3, 0.5, 0.7) Medium	(0.6, 0.8, 1) High	(0.8, 1 1) Very High
Q1	Econ	✔				
Q2	Tech			✔		

Figure A.1. Questionnaire form used to facilitate importance of main attributes with respect to the overall goal

With respect to the main attribute Economic Attributes

Q3. What degree of importance do you assign to the sub-attribute Investment Costs (InvC)?

Q4. What degree of importance do you assign to the sub-attribute Operating Costs (OC)?

With respect to: Economic Attributes		Importance of one sub-attribute with respect to main attribute Research				
Questions	Attributes	(0, 0, 0.2) Very Low	(0, 0.2, 0.4) Low	(0.3, 0.5, 0.7) Medium	(0.6, 0.8, 1) High	(0.8, 1 1) Very High
Q3	InvC			✔		
Q4	OC	✔				

Figure A. 2. Questionnaire forms used to facilitate importance of sub-attributes with respect to main attributes

With respect to the sub-attribute Investment Costs (InvC)

Q5. What degree of importance do you assign to the sub-attribute Special Tooling (ST)?

Q6. What degree of importance do you assign to the sub-attribute Installation costs (InsC)?

Q7. What degree of importance do you assign to the sub-attribute Purchase cost (PC)?

With respect to: Service		Importance of one sub-attribute with respect to main attribute Service				
Questions	Sub-Attributes	(0, 0, 0.2) Very Low	(0, 0.2, 0.4) Low	(0.3, 0.5, 0.7) Medium	(0.6, 0.8, 1) High	(0.8, 1 1) Very High
Q5	ST	✔				
Q6	InsC					✔
Q7	PC			✔		

Figure A.3. Questionnaire forms used to facilitate importance of sub-sub-attributes with respect to sub-attributes

Scoring of Alternatives with respect to sub-sub-attributes

Q8. What scores do you assign to each Industrial Robotic System with respect to the sub-sub-attribute Installation costs (InsC)?

Q9. What scores do you assign to each Industrial Robotic System with respect to the sub-sub-attribute Special Tooling (ST)?

Q10. What scores do you assign to each Industrial Robotic System with respect to the sub-sub-attribute Purchase cost (PC)?

Q11. What scores do you assign to each Industrial Robotic System with respect to the sub-sub-attribute Maintenance Costs (MaC)?

Q12. What scores do you assign to each Industrial Robotic System with respect to the sub-sub-attribute Labor Costs (LC)?

Q13. What scores do you assign to each Industrial Robotic System with respect to the sub-sub-attribute Training Costs (TC)?

Questions	Attributes	Alternatives	Very Low (0, 0, 20)	Low (0, 20, 40)	Fair (30, 50, 70)	High (60, 80, 100)	Very High (80, 100, 100)
Q8	InsC	AH-1		✔			
		AH-2	✔				
		AH-3			✔		
Q9	ST	AH-1				✔	
		AH-2		✔			
		AH-3			✔		
…	…	…	…	…	…	…	…
Q13	TC	AH-1					✔
		AH-2		✔			
		AH-3			✔		

Figure A.4. Questionnaire form used to facilitate scores of alternatives with respect to sub- and sub-sub-attributes

APPENDIX B

Table B.1. r_{ij}

	AH-1	AH-2	AH-3
PC	(0.191, 0.5, 1.244)	(0.506, 1, 1.978)	(0.236, 0.359, 1.022)
InsC	(0.506, 1, 1.978)	(0.36, 0.577, 1.689)	(0.146, 0.333, 1.267)
STF	(0.371, 0.717, 1.171)	(0.613, 0.83, 1.195)	(0.661, 1, 1.512)
Mac	(0.161, 0.396, 0.878)	(0.468, 0.736, 1.268)	(0.661, 1, 1.512)
LC	(0.54, 1, 1.853)	(0.397, 0.647, 1.088)	(0.143, 0.333, 0.971)
TC	(0.171, 0.933, 2.094)	(0.421, 1, 2.375)	(0.079, 0.489, 1.438)
SP	(0.296, 1, 2.793)	(0.16, 0.63, 1.931)	(0.358, 0.667, 1.655)
Wl	(0.211, 0.689, 1.438)	(0.303, 0.778, 1.75)	(0.421, 1, 2.375)
Pro	(0.568, 1, 1.723)	(0.58, 0.884, 1.617)	(0.358, 0.522, 1.021)
NA	(0.185, 0.508, 1.103)	(0.111, 0.333, 1)	(0.481, 1, 2.077)
Rep	(0.582, 1, 1.656)	(0.491, 0.773, 1.719)	(0.273, 0.727, 1.344)
Pre	(0.153, 0.672, 2)	(0.247, 0.836, 2.167)	(0.424, 1, 2.361)
MeC	(0.644, 0.903, 1.393)	(0.195, 0.444, 0.518)	(0.506, 1, 1.554)

Table B.2. v_{ij}

	AH-1	AH-2	AH-3
PC	(0.015, 0.223, 0.857)	(0.039, 0.447, 1.362)	(0.018, 0.16, 0.704)
InsC	(0.033, 0.304, 1.159)	(0.024, 0.175, 0.989)	(0.01, 0.101, 0.742)
STF	(0.019, 0.172, 0.64)	(0.031, 0.2, 0.653)	(0.034, 0.241, 0.826)
Mac	(0.018, 0.156, 0.57)	(0.051, 0.291, 0.824)	(0.072, 0.395, 0.982)
LC	(0.057, 0.446, 1.302)	(0.042, 0.288, 0.764)	(0.015, 0.149, 0.682)
TC	(0.006, 0.265, 1.123)	(0.015, 0.284, 1.274)	(0.003, 0.139, 0.771)
SP	(0.051, 0.54, 1.946)	(0.028, 0.34, 1.345)	(0.062, 0.36, 1.153)
Wl	(0.042, 0.382, 1.063)	(0.061, 0.431, 1.294)	(0.085, 0.554, 1.757)
Pro	(0.082, 0.376, 1.097)	(0.084, 0.333, 1.029)	(0.052, 0.196, 0.65)
NA	(0.009, 0.148, 0.531)	(0.006, 0.097, 0.482)	(0.025, 0.291, 1)
Rep	(0.039, 0.398, 1.011)	(0.033, 0.307, 1.049)	(0.018, 0.289, 0.82)
Pre	(0.017, 0.372, 1.479)	(0.028, 0.463, 1.602)	(0.047, 0.554, 1.746)
MeC	(0.084, 0.436, 1.042)	(0.026, 0.215, 0.387)	(0.066, 0.483, 1.162)

Table B.3. $M(v_{ij})$

	AH-1	AH-2	AH-3
PC	0.365	0.616	0.294
InsC	0.498	0.396	0.284
STF	0.276	0.294	0.366
Mac	0.248	0.388	0.483
LC	0.601	0.364	0.281
TC	0.464	0.524	0.304
SP	0.845	0.570	0.524
Wl	0.495	0.595	0.798
Pro	0.518	0.481	0.299
NA	0.229	0.194	0.438
Rep	0.482	0.463	0.375
Pre	0.622	0.697	0.782
MeC	0.5208158	0.209	0.570

Table B.4. D_{ij}^{*}

	AH-1	AH-2	AH-3
PC	0.214	0.000	0.300
InsC	0.000	0.118	0.222
STF	0.101	0.061	0.000
Mac	0.323	0.121	0.000
LC	0.000	0.181	0.322
TC	0.0167	0.000	0.160
SP	0.000	0.133	0.140
Wl	0.149	0.092	0.000
Pro	0.000	0.044	0.240
NA	0.220	0.298	0.000
Rep	0.000	0.082	0.121
Pre	0.112	0.055	0.000
MeC	0.045	0.455	0.000

FUZZY MULTI-ATTRIBUTE SCORING METHODS WITH APPLICATIONS

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Abstract: The multi-attribute scoring methods are widely used while comparing the alternatives because of their simplicity. In the case of incomplete information and vagueness, these multi-attribute scoring methods have been extended to obtain the fuzzy versions. In this chapter, fuzzy simple additive weighting methods and fuzzy multiplicative weighting methods are explained with numerical examples.

Key words: Scoring, simple additive weighting, multiplicative weighting, multi-attribute

1. INTRODUCTION

An index formulation of a system when the decision maker has a thorough understanding of the functional relationships among its components, or when he or she possesses sufficient data to regress a statistical relationship, can be used in modeling a multi-attribute problem. Since it often cannot be expected that any of these conditions will be met easily in a normal decision-making environment, this chapter presents two scoring techniques: the Simple Additive Weighting method, which obtains an index by adding contributions from each attribute, and the Weighted Product method, which obtains the index by multiplying contributions from attributes.

1.1 Crisp Simple Additive Weighting (CSAW) Method

The SAW method is probably the best known and most widely used multiple attribute decision-making (MADM) method. A score in the SAW

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method is obtained by adding contributions from each attribute. Since two items with different measurement units cannot be added, a common numerical scaling system such as normalization is required to permit addition among attribute values. The total score for each alternative then can be computed by multiplying the comparable rating for each attribute by the importance weight assigned to the attribute and then summing these products over all attributes (Yoo and Hwang, 1995).

Formally the value of an alternative in the SAW method can be expressed as

$$V\left(A_i\right) = V_i = \sum_{j=1}^{n} w_j v_j\left(x_{ij}\right), \; i = 1,2,...,m \tag{1}$$

where $V(A_i)$ is the value function of alternative A_i and w_j and v_j (.) are weight and value functions of attribute X_j, respectively. Or the performance of alternative A_i is calculated by

$$V\left(A_i\right) = \frac{\sum_{j=1}^{n} w_j r_{ij}}{\sum_{j=1}^{n} w_j} \tag{2}$$

where r_{ij} is the rating of the ith alternative under the jth attribute with a numerically comparable scale.

Through the normalization process, each incommensurable attribute becomes a pseudo-value function, which allows direct addition among attributes. The value of alternative A_i can be rewritten as

$$V_i = \sum_{j=1}^{n} w_j r_{ij}, \qquad i = 1,2,...,m \tag{3}$$

where r_{ij} is the comparable scale of x_{ij}, which can be obtained by a normalization process. The underlying assumption of the SAW method is that attributes are preferentially independent. Less formally, this means that the contribution of an individual attribute to the total (multiattribute) score is independent of other attribute values. Therefore, the decision maker's preference (or feelings) regarding the value of one attribute are not influenced in any way by the values of the other attributes (Fishburn, 1976). Fortunately, studies (Edwards, 1977; Farmer, 1987) show that the

SAW method yields extremely close approximations to "true" value functions even when independence among attributes does not exactly hold.

In addition to the preference independence assumption, the SAW has a required characteristic for weights. That is, the SAW presumes that weights are proportional to the relative value of a unit change in each attribute's value function (Hobbs, 1980). For instance, let us consider a value function with two attributes: $V = w_1v_1 + w_2v_2$. By setting the amount of V constant, we can derive the relationship of $w_1/w_2 = -\Delta v_2/\Delta v_1$. This relationship indicates that if $w_1 = 0.33$ and $w_2 = 0.66$, the decision maker must be indifferent to the trade between two units of v_1 and one unit of v_2.

1.2 Crisp Weighted Product (CWP) Method

In the SAW method, addition among attribute values was allowed only after the different measurement units were transformed into a dimensionless scale by a normalization process. However, this transformation is not necessary if attributes are connected by multiplication. When we use multiplication among attribute values, the weights become exponents associated with each attribute value, a positive power for benefit attributes, and a negative power for cost attributes. Formally, the value of alternative A_i is given by (Yoo and Hwang, 1995)

$$V\left(A_i\right) = V_i = \prod_{j=1}^{n} x_{ij}^{w_j}, \quad i = 1, 2, \ldots, m \tag{4}$$

Because of the exponent property, this method requires that all ratings be greater than 1. For instance, when an attribute has fractional ratings, all ratings in that attribute are multiplied by 10^m to meet this requirement. Alternative values obtained by the multiplicative method do not have a numerical upper bound. The decision maker may also not find any true meaning in those values. Hence, it may be convenient to compare each alternative value with the standard value. If we use the ideal alternative A^* for the comparison purpose, the ratio between an alternative and the ideal alternative is given by

$$R_i = \frac{V\left(A_i\right)}{V\left(A^*\right)} = \frac{\prod_{j=1}^{n} x_{ij}^{w_j}}{\prod_{j=1}^{n} \left(x_j^*\right)^{w_j}}, \quad i = 1, 2, \ldots, m \tag{5}$$

where x_i^* is the most favorable value for the jth attribute. It is clear that $0 \le R_i \le 1$ and the preference of A_i increases when R_i approaches.

2. FUZZY SETS AND FUZZY NUMBERS

To deal with vagueness of human thought, Zadeh [1] first introduced the fuzzy set theory, which was based on the rationality of uncertainty due to imprecision or vagueness. A major contribution of fuzzy set theory is its capability of representing vague knowledge. The theory also allows mathematical operators and programming to apply to the fuzzy domain.

A fuzzy number is a normal and convex fuzzy set with membership function $\mu_A(x)$, which both satisfies normality: $\mu_A(x)=1$, for at least one $x \in R$, and convexity:

$$\mu_A(x') \ge \mu_A(x_1) \Lambda \mu_A(x_2)$$

where $\mu_A(x) \in [0,1]$ and $\forall x' \in [x_1, x_2]$. "Λ" stands for the minimization operator.

The definition of a triangular fuzzy number has been given in Chapter 4. A flat fuzzy number (FFN) is shown in Figure 1. The membership function of a FFN, $\tilde{V}$ is defined by

$$\mu(x|\tilde{V}) = (m_1, f_1(y|\tilde{V}) / m_2, m_3 / f_2(y|\tilde{V}), m_4) \tag{6}$$

where $m_1 \prec m_2 \prec m_3 \prec m_4$, $f_1(y|\tilde{V})$ is a continuous monotone increasing function of y for $0 \le y \le 1$ with $f_1(0|\tilde{V}) = m_1$ and $f_1(1|\tilde{V}) = m_2$ and $f_2(y|\tilde{V})$ is a continuous monotone decreasing function of y for $0 \le y \le 1$ with $f_2(1|\tilde{V}) = m_3$ and $f_2(0|\tilde{V}) = m_4$. $\mu(y|\tilde{V})$ is denoted simply as $(m_1 / m_2, m_3 / m_4)$.

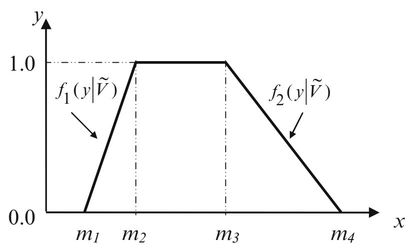

Figure 1. A flat fuzzy number, V

3. FUZZY SCORING METHODS

Zhang et al. (2001) applied fuzzy logic to compute proximity between an intellectual property query and a specification, which are tree-structured models constructed from their respective Extensible Markup Language representation. Bector et al. (2002) derived a formula for fuzzy scoring model assuming that both the weights and the ratings were fuzzy and demonstrated the use of the main formula with a numerical example. Lo (2002) proposed an operating mechanism based on fuzzy theory to integrate fuzzy composite scores of multiple assessments and applied simulation to test this fuzzy scoring frame. Mitra et al. (2002) described a fuzzy knowledge-based network based on the linguistic rules using the principle of a fuzzy decision tree. They demonstrated the effectiveness of the system on three sets of real-life data.

Kwong et al. (2002) introduced a scoring method combined with a fuzzy expert system in supplier assessment and evaluated their system for the supplier assessment of electrical appliance products. Belacel and Boulassel (2004) proposed a new multi-criteria fuzzy classification procedure called PROCFTN, and they also tested this method with an experimental set of 250 cases of astrocytic tumors. Ohdar and Kumar (2004) proposed a fuzzy system to evaluate the suppliers' performance. They developed a Genetic Algorithm-based methodology to evolve the optimal set of a fuzzy rule-based system and used a fuzzy inference system of the MATLAB fuzzy logic toolbox to assess the suppliers' performance. Zhang (2004) presented the necessity to use fuzzy data for a handover decision in heterogonous networks and provided new handover criteria along with a new handover decision strategy. Graf (2005) proposed a game scoring system with the score submission and ranking component, and when compared with statistical approach showed that the fuzzy logic approach was more adequate to build into scorings.

3.1 Fuzzy Simple Additive Weighting Methods with Numerical Examples

The crisp simple additive weighting method explained above can be transformed to the fuzzy case as follows:

When both w_j and r_{ij} are fuzzy sets

$$w_j = \{(y_j, \mu_{wj}(y_j))\}, \forall j \tag{7}$$

and

$$r_{ij} = \{(x_{ij}, \mu_{rij}(x_{ij}))\}, \forall i, j \tag{8}$$

where y_j and x_{ij} take their numbers on the real line $\Re$ and $\mu_{wj}(y_j)$ and $\mu_{rij}(x_{ij})$ take values in [0, 1]. The utility of alternative A_i, $u_i = \{(u_i, \mu_{ui}(u_i))\}$, can be calculated as follows:

The variable u_i takes its value on the real line $\Re$ and can be obtained using

$$u_i = \frac{\sum_{j=1}^{n} y_j x_{ij}}{\sum_{j=1}^{n} y_j} \tag{9}$$

The membership function $\mu_{ui}(u_i)$ can be calculated using

$$\mu_{ui}(u_i) = \sup_{\vee} \left\{ \left[\bigwedge_{j=1}^{n} \mu_{wj}(y_j) \right] \wedge \left[\bigwedge_{j=1}^{n} \mu_{rij}(x_{ij}) \right] \right\} \tag{10}$$

where $\vee = (y_1, \ldots, y_n, x_{i1}, \ldots, x_{in})$

The membership function $\mu_{ui}(u_i)$ is not directly obtainable when $\mu_{wj}(y_j)$ and $\mu_{rij}(x_{ij})$ are piecewise continuously differentiable functions. To resolve this difficulty and preserve the simplicity of the simple additive weighting method, several approaches have been proposed (Chen et al., 1992). Some of them are explained in the following discussion. The first four approaches use the $\alpha - cut$ to approximate the $\mu_{ui}(u_i)$. The fifth one, Bonissone's approach, assumes that all piecewise continuously differentiable fuzzy numbers can be approximated by L-R-type trapezoidal numbers.

3.1.1 Baas and Kwakernaak's Approach

It is assumed that $\mu_{wj}(y_j)$ and $\mu_{rij}(x_{ij})$ are normalized membership functions. The approximate fuzzy utility U_i for alternative A_i is determined using the following steps:

Step 1. Set an α_0 level for $\mu_{ui}(u_i)$.

Step 2. Identify the y_j and x_{ij} values that satisfy

$$\mu_{wj}\left(y_j\right)=\mu_{rij}\left(x_{ij}\right)=\alpha_0, \quad \forall i, j \tag{11}$$

Step 3. There are many u_i values such that $\mu_{ui}\left(u_i\right)=\alpha_0$ and the extreme ones, u_{imin} and u_{imax}, must be determined. Given a set of real numbers $\left(\hat{y}_1,\cdots,\hat{y}_n,\hat{x}_{i1},\cdots,\hat{x}_{in}\right)$ such that $\mu'_{r_i}\left(\hat{x}_{ij}\right)$ and $\mu'_{wj}\left(\hat{y}_j\right)/\left(\hat{x}_{ij}-u_i\right), \quad \forall i, j$, where

$$\mu'_{ri}\left(x_{ij}\right)=d\mu_{ri}\left(x_{ij}\right)/dx_{ij} \tag{12}$$

and

$$\mu'_{wj}\left(y_j\right)=d\mu_{wj}\left(y_j\right)/dy_j \tag{13}$$

have the same sign. The resulting u_i will be either u_{imax} or u_{imin}.

A Numerical Example

Assume that we have a decision matrix as follows.

$$\begin{matrix} & X1 & X2 \\ \begin{matrix} A1 \\ A2 \end{matrix} & \multicolumn{2}{c}{\begin{bmatrix} \text{excellent} & \text{fair} \\ \text{good} & \text{good} \end{bmatrix}} \end{matrix}$$

where $\left(r_{11},r_{12}\right)=\left(\text{excellent, fair}\right)$ and $\left(r_{21},r_{22}\right)=\left(\text{good, good}\right)$. Let the weight set be $\left(w_1,w_2\right)=\left(\text{important, very important}\right)$. Figure 2 represents these linguistic terms.

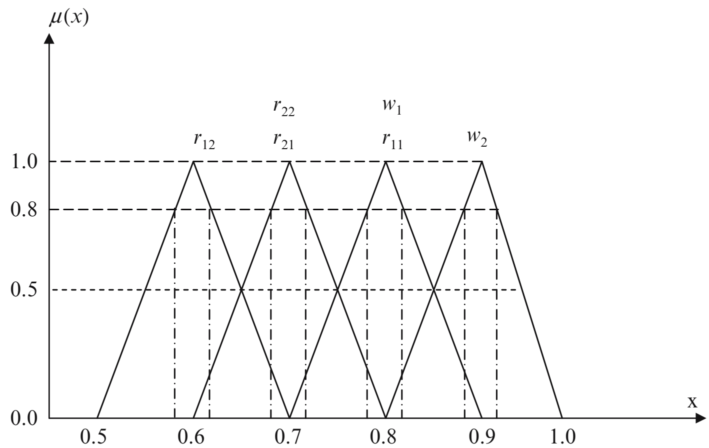

Figure 2. Fuzzy representation of linguistic terms

Step 1. Let's set $\alpha_0 = 0.80$.

Step 2. Let's identify $\hat{x}_{11}, \hat{x}_{12}, \hat{x}_{21}, \hat{x}_{22}, \hat{y}_1, \hat{y}_2$ values such that $\mu_{r11}(\hat{x}_{11}) = \mu_{r12}(\hat{x}_{12}) = \mu_{r21}(\hat{x}_{21}) = \mu_{w1}(\hat{y}_1) = \mu_{w2}(\hat{y}_2) = 0.80$. The values providing this equality are given in the Table 1.

Table 1. $\alpha - cut$ Values While $\alpha_0 = 0.80$

$\hat{x}_{12}$	$\hat{x}_{11}$	$\hat{x}_{21}$	$\hat{x}_{22}$	$\hat{y}_1$	$\hat{y}_2$
0.62	0.82	0.72	0.72	0.82	0.92
0.58	0.78	0.68	0.68	0.78	0.88

Step 3. There are a total of 2^4 = 16 possible combinations of $(\hat{x}_{11}, \hat{x}_{12}, \hat{y}_1, \hat{y}_2)$ and $(\hat{x}_{21}, \hat{x}_{22}, \hat{y}_1, \hat{y}_2)$. Using Eq. (9) on all x_{ij} and y_j combinations, we obtain 16 u_1 values. The u_1 values are given in Table 2.

From Table 2, $\mu_{U1}(u_1 = 0.716471) = 0.80$ and $\mu_{U1}(u_1 = 0.671765) = 0.80$. To calculate u_2 values for $\alpha_0 = 0.80$, the similar operations are made and the following Table 3 is obtained:

Table 2. Possible Combinations of x_{ij} and y_j and Their Corresponding u_1 Values

x_{11}	x_{12}	y_1	y_2	u_1	
0.82	0.62	0.82	0.92	0.714253	
0.82	0.62	0.82	0.88	0.716471	MAX.
0.82	0.62	0.78	0.92	0.711765	
0.82	0.62	0.78	0.88	0.713976	
0.82	0.58	0.82	0.92	0.693103	
0.82	0.58	0.82	0.88	0.695765	
0.82	0.58	0.78	0.92	0.690118	
0.82	0.58	0.78	0.88	0.692771	
0.78	0.62	0.82	0.92	0.695402	
0.78	0.62	0.82	0.88	0.697176	
0.78	0.62	0.78	0.92	0.693412	
0.78	0.62	0.78	0.88	0.695181	
0.78	0.58	0.82	0.92	0.674253	
0.78	0.58	0.82	0.88	0.676471	
0.78	0.58	0.78	0.92	0.671765	MIN.
0.78	0.58	0.78	0.88	0.673976	

Table 3. Possible Combinations of x_{ij} and y_j and Their Corresponding u_2 Values

x_{21}	x_{22}	y_1	y_2	u_2	
0.72	0.72	0.82	0.92	0.720000	MAX.
0.72	0.72	0.82	0.88	0.720000	MAX.
0.72	0.72	0.78	0.92	0.720000	MAX.
0.72	0.72	0.78	0.88	0.720000	MAX.
0.72	0.68	0.82	0.92	0.698851	
0.72	0.68	0.82	0.88	0.699294	
0.72	0.68	0.78	0.92	0.698353	
0.72	0.68	0.78	0.88	0.698795	
0.68	0.72	0.82	0.92	0.701149	
0.68	0.72	0.82	0.88	0.700706	
0.68	0.72	0.78	0.92	0.701647	
0.68	0.72	0.78	0.88	0.701205	
0.68	0.68	0.82	0.92	0.680000	MIN.
0.68	0.68	0.82	0.88	0.680000	MIN.
0.68	0.68	0.78	0.92	0.680000	MIN.
0.68	0.68	0.78	0.88	0.680000	MIN.

For $\alpha_0 = 0.0;\ 0.50;$ and 1.00 values, Tables 4–6 is obtained.

Table 4. $\alpha - cut$ Values While $\alpha_0 = 0.00$

$\hat{x}_{12}$	$\hat{x}_{11}$	$\hat{x}_{21}$	$\hat{x}_{22}$	$\hat{y}_1$	$\hat{y}_2$
0.70	0.50	0.60	0.60	0.70	0.80
0.90	0.70	0.80	0.80	0.90	1.00

Table 5. $\alpha - cut$ Values While $\alpha_0 = 0.50$

$\hat{x}_{12}$	$\hat{x}_{11}$	$\hat{x}_{21}$	$\hat{x}_{22}$	$\hat{y}_1$	$\hat{y}_2$
0.75	0.55	0.65	0.65	0.75	0.85
0.85	0.65	0.75	0.75	0.85	0.95

Table 6. $\alpha - cut$ Values While $\alpha_0 = 1.00$

$\hat{x}_{12}$	$\hat{x}_{11}$	$\hat{x}_{21}$	$\hat{x}_{22}$	$\hat{y}_1$	$\hat{y}_2$
0.80	0.60	0.70	0.70	0.80	0.90
0.80	0.60	0.70	0.70	0.80	0.90

The final utility results are found for u_1 as follows:

Table 7. The Utility Values of u_1

$\mu_{u1}(u_1 = \alpha_0)$	0	0.50	0.80	$\mu_{u1}(u_1 = \alpha_0)$
u_{1max}.	0.805882	0.744444	0.716471	0.694118
u_{1min}	0.582353	0.638235	0.671765	0.694118

The final utility results are found for u_2 as follows:

Table 8. The Utility Values of u_2

$\mu_{u1}(u_1 = \alpha_0)$	0	0.50	0.80	$\mu_{u1}(u_1 = \alpha_0)$
u_{1max}.	0.80	0.75	0.72	0.70
u_{1min}	0.60	0.65	0.68	0.70

The results in Tables 7 and 8 can be represented in Figure 3:

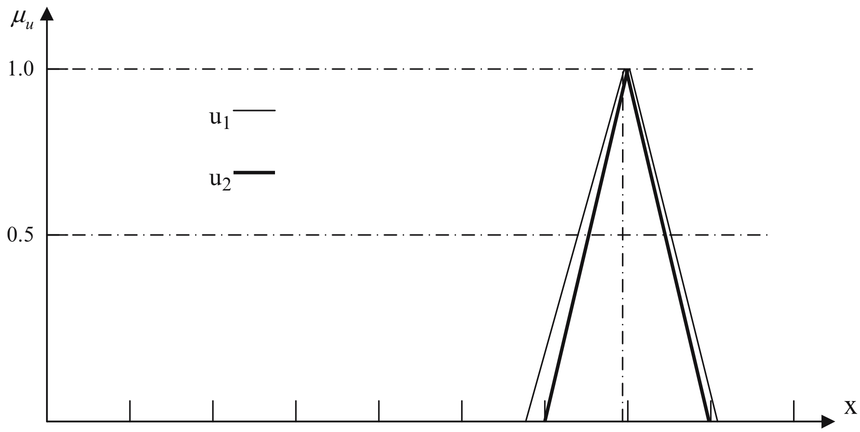

Figure 3. The alternatives' fuzzy utilities using Baas and Kwakernaak's approach (1977)

The ranking of u_1 and u_2 can be made by using a proper ranking method. Here, it is clear that $u_2 \succ u_1$.

3.1.2 Kwakernaak's Approach (1979)

This approach is a modification of Baas and Kwakernaak's (1977) approach. This approach proposes an improved algorithm to find the maximum and minimum u_i values instead of selecting them among all the possible values. For more details, see Chen et al. (1992).

3.1.3 Dubois and Prade's Approach (1982)

Dubois and Prade (1982) proposed a more efficient search procedure to obtain u_i values. This approach assumes that all fuzzy weights w_j and fuzzy r_{ij} are normalized fuzzy numbers. The α–level sets are used to derive fuzzy utilities based on the classic SAW method. The steps of this approach are given in the following:

Step 1. Set an α level and determine α – level sets for w_j and r_{ij} to be

$$w_{j\alpha} = \left[y_j^-, y_j^*\right] \quad \forall j \tag{14}$$

$$r_{ij\alpha} = \left[x_{ij}^-, x_{ij}^*\right] \quad \forall i, j \tag{15}$$

Step 2. Compute the normalized fuzzy weights, $P_j, \forall j$. When the α –level sets of w_j are known, n α – level sets of the normalized fuzzy weights P_j, $\forall j$, can be obtained:

$$P_j^* = y_j^* \Big/ \left(y_j^* + \sum_{k=j} y_k^- \right) \tag{16}$$

and

$$P_j^- = y_j^- \Big/ \left(y_j^- + \sum_{k=j} y_k^* \right) \tag{17}$$

Let $q_j \in \left[p_j^-, p_j^*\right] \forall j$. Then $\sum_{j=1}^{n} q_j = 1$.

Step 3. For alternative A_i, the rating r_{ij} may be represented by an α – level set as in Eq. (15). To order x_{ij}^- and x_{ij}^*, $\forall j$ as

$$m_1^- \le m_2^- \le \ldots \le m_n^- \tag{18}$$

in which $m_1^- = \min_j x_{ij}^-$ and $m_n^- = \max_j x_{ij}^-$ and

$$m_1^* \le m_2^* \le \ldots \le m_n^* \tag{19}$$

in which $m_1^* = \min_j x_{ij}^*$ and $m_n^* = \max_j x_{ij}^*$.

Step 4. The smallest upper and the largest lower bound of u_i are computed as

$$u_{imin} = \left(\sum_{j=1}^{d-1} p_j^* m_j^- \right) + \left[1 - \sum_{j=1}^{d-1} p_j^* - \sum_{j=d+1}^{n} p_j^- \right] m_d^- + \sum_{j=d+1}^{n} p_j^- m_j^- \tag{20}$$

$$u_{imax} = \left(\sum_{j=1}^{e-1} p_j^- m_j^* \right) + \left[1 - \sum_{j=1}^{e-1} p_j^- - \sum_{j=e+1}^{n} p_j^* \right] m_e^* + \sum_{j=e+1}^{n} p_j^* m_j^* \tag{21}$$

The unknown parameters in Eqs. (20) and (21) are d and e. The parameter d is determined when the following equality is satisfied:

$$1 - \sum_{j=1}^{d-1} p_j^* - \sum_{j=d+1}^{n} p_j^- = z_d \in \left[p_d^-, p_d^* \right] \tag{22}$$

Similarly, the parameter e is determined when the following equality is satisfied:

$$1 - \sum_{j=1}^{e-1} p_j^- - \sum_{j=e+1}^{n} p_j^* = z_e \in \left[p_e^-, p_e^* \right] \tag{23}$$

Step 5. The fuzzy utility u_i can be represented by the interval $[u_{imin}, u_{imax}]$ at any α level. The decision maker can set several α levels and repeat the algorithm several times to derive an approximated fuzzy utility u_i.

3.1.4 Cheng and McInnis's (1980) Approach

Cheng and McInnis (1980) pointed out that continuous membership functions of r_{ij} and w_j are the cause of the complexity of obtaining fuzzy utilities. To avoid such difficulty, they suggested first to convert the continuous membership function to discrete ones and then compute the fuzzy utilities using the following algorithm:

Step 1. The continuous membership function is converted to a discrete one. This is done by having the decision maker specify the number of α levels that he/she wants to use. The width of intervals is determined according to the decision maker's preference. The decision maker may specify different numbers of α levels and widths of intervals for different membership functions in a MCDM problem.

Step 2. For each $\alpha-$level, the steps 3 and 4 are performed. The first $\alpha-$level to be considered is the largest one among all the w_j *'s* and r_{ij} *'s*.

Step 3. Given α_0, the $\alpha-$level set for each r_{ij} and w_j can be obtained as:

$$r_{ij\alpha_0} = \left[x_{ij}^-, x_{ij}^*\right], \quad \forall i, j \tag{24}$$

and

$$w_{j\alpha_0} = \left[y_j^-, y_j^*\right], \quad \forall j \tag{25}$$

Step 4. Given the upper and lower bounds of r_{ij} and w_j at the α_0–level, the upper and lower bounds of the fuzzy utility at α_0, $u_{i\alpha 0} = \left[u_{imin}, u_{imax}\right]$, can be computed by following the sub-steps:

Step 4.1. Compute u_{imax} using the upper bound of r_{ij}, $\forall j$, i.e., x_{ij}^*:

$$u_i = \frac{\sum_j y_j x_{ij}}{\sum_j y_j} \tag{26}$$

To maximize u_i, it must be decided whether y_j^- or y_j^* should be used.

Step 4.2. After finding u_{imax}, u_{imin} can be easily identified. First, x_{ij}^- is used for all r_{ij}. Second, for those w_j whose upper bounds were used for deriving u_{imax}, their lower bounds in computing u_{imin} will be used and vice versa.

Steps 3 and 4 are used for the next largest $\alpha-$level until all α levels are exhausted.

3.1.5 Bonissone's (1982) Approach

Bonissone (1982) assume that fuzzy/crisp information in decision problems can be approximated by a parameter-based representation. It is called the L-R-type trapezoidal number (see Figure 4). Fuzzy arithmetic operations with L-R type trapezoidal numbers are given in the following discussion. Let $\tilde{M} = (a, b, \alpha, \beta)$ and $\tilde{N} = (c, d, \gamma, \delta)$ be positive trapezoidal fuzzy numbers:

$$\tilde{M} + \tilde{N} = (a + c, b + d, \alpha + \gamma, \beta + \delta) \tag{27}$$

$$\tilde{M} - \tilde{N} = (a - d, b - c, \alpha + \delta, \beta + \gamma) \tag{28}$$

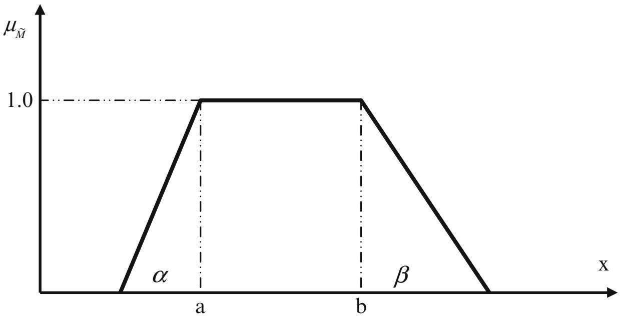

Figure 4. L-R-Type trapezoidal fuzzy number, $\tilde{M} = (a, b, \alpha, \beta)$

$$\tilde{M} \times \tilde{N} = (ac, bd, a\gamma + c\alpha - \alpha\gamma, b\delta + d\beta + \beta\delta) \tag{29}$$

$$\tilde{M} \div \tilde{N} = \left(\frac{a}{d}, \frac{b}{c}, \frac{a\delta + d\alpha}{d(d+\delta)}, \frac{b\gamma + c\beta}{c(c-\gamma)} \right). \tag{30}$$

Using the algebraic operations above, one can easily compute the performance of an alternative with respect to the attributes by using:

$$u_i = \sum_{j=1}^{n} w_j r_{ij} \tag{31}$$

where w_j and r_{ij} may be crisp or fuzzy numbers represented in the *L-R* trapezoidal number format.

A Numerical Example

Three alternatives of advanced manufacturing systems, FMS-1, FMS-2, and FMS-3, will be evaluated with respect to four attributes: engineering effort (X_1), flexibility (X_2), net present worth (X_3), and integration ability (X_4). The decision matrix is given as

D =

	X_1	X_2	X_3	X_4
FMS_1	fair	good	fair	good
FMS_2	fair	very good	bad	good
FMS_3	very bad	very good	very good	very bad

The weight vector is given as $\widetilde{w} = \{\text{important, more or less important, unimportant, very important}\}$; where very unimportant: (0, 0.2, 0, 0.1); unimportant: (0.3, 0.3, 0.1, 0.1); more or less unimportant: (0.4, 0.4, 0.1, 0.1); indifferent: (0.5, 0.5, 0.1, 0.1); more or less important: (0.6, 0.6, 0.1, 0.1); important: (0.7, 0.7, 0.1, 0.1); very important: (0.8, 0.8, 0.1, 0.2).

The fuzzy set associated with each linguistic term is as follows: very bad: (0, 0.2, 0, 0.1); bad: (0.3, 0.3, 0.1, 0.1); more or less bad: (0.4, 0.4, 0.1, 0.1); fair: (0.5, 0.5, 0.1, 0.1); more or less good: (0.6, 0.6, 0.1, 0.1); good: (0.7, 0.7, 0.1, 0.1); very good: (0.8, 0.8, 0.1, 0.2).

Then, the fuzzy utilities for the alternatives are computed as follows:

$$\begin{aligned} U_1 = \sum_{j=1}^{4} \tilde{w}_j \tilde{x}_{1j} &= (0.7,\ 0.7,\ 0.1, 0.\ 1) \otimes (0.5, 0.5, 0.1, 0.1) \oplus (0.6, 0.6, 0.1, 0.1) \\ &\otimes (0.7, 0.7, 0.1, 0.1) \oplus (0.3, 0.3, 0.1, 0.1) \otimes (0.5, 0.5, 0.1, 0.1) \\ &\oplus (0.8, 0.8, 0.1, 0.2) \otimes (0.7, 0.7, 0.1, 0.1) \\ &= (1.48,\ 1.48,\ 0.44,\ 0.52) \end{aligned}$$

$$\begin{aligned} U_2 = \sum_{j=1}^{4} \tilde{w}_j \tilde{x}_{2j} &= (0.7,\ 0.7,\ 0.1,\ 0.1) \otimes (0.5, 0.5, 0.1, 0.1) \oplus (0.6, 0.6, 0.1, 0.1) \\ &\otimes (0.8, 0.8, 0.1, 0.2) \oplus (0.3, 0.3, 0.1, 0.1) \otimes (0.3, 0.3, 0.1, 0.1) \\ &\oplus (0.8, 0.8, 0.1, 0.2) \otimes (0.7, 0.7, 0.1, 0.1) \\ &= (1.48,\ 1.48,\ 0.43,\ 0.66) \end{aligned}$$

$$\begin{aligned} U_3 = \sum_{j=1}^{4} \tilde{w}_j \tilde{x}_{3j} &= (0.7,\ 0.7,\ 0.1,\ 0.1) \otimes (0, 0.2, 0, 0.1) \oplus (0.6, 0.6, 0.1, 0.1) \\ &\otimes (0.8, 0.8, 0.1, 0.2) \oplus (0.3, 0.3, 0.1, 0.1) \otimes (0.8, 0.8, 0.1, 0.2) \\ &\oplus (0.8, 0.8, 0.1, 0.2) \otimes (0, 0.2, 0, 0.1) \\ &= (0.72,\ 1.02,\ 0.25,\ 0.62) \end{aligned}$$

The obtained fuzzy utilities are illustrated in Figure 5.

The ranking of $U_1, U_2,$ and U_3 can be made by using any ranking method. In this example, it is clear that U_2 is the largest fuzzy number. Thus, FMS_2 is selected.

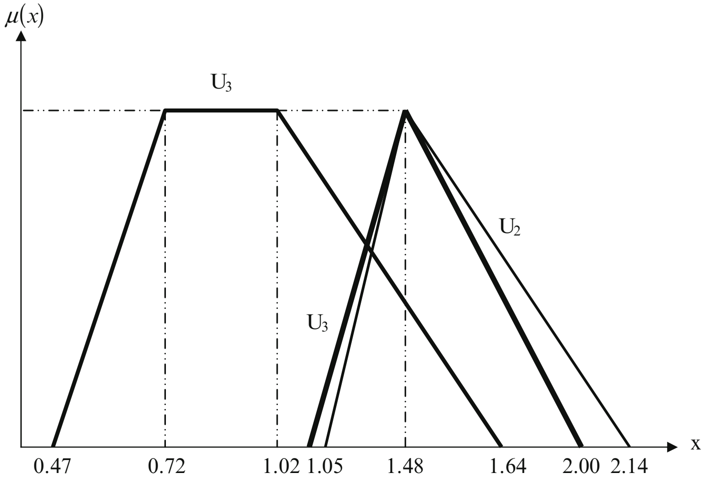

Figure 5. The fuzzy utilities of the example problem

3.1.6 Bector et al.'s Approach (2002)

Bector et al.'s approach (2002) is a direct treatment with fuzzy numbers to the crisp case. In this approach, each w_i is represented by a triangular fuzzy number given as $w_i = (w_{i1}, w_{i2}, w_{i3})$, i = 1, 2, . . . , m, whose $\alpha-$ cut is given by

$$w_i(\alpha) = [w_{i1} + (w_{i2} - w_{i1})\alpha, w_{i3} - (w_{i3} - w_{i2})\alpha] \tag{32}$$

and each v_{ij} is represented by a triangular fuzzy number given as $v_{ij} = (v_{ij1}, v_{ij2}, v_{ij3})$, j=1,2,…, n, whose $\alpha-$ cut is given by

$$V_i(\alpha) = \sum_{i=1}^{m} w_i(\alpha) \otimes v_{ij}(\alpha) \tag{33}$$

$$V_i(\alpha) = \left(\sum_{i=1}^{m} w_{i1} v_{ij1}, \sum_{i=1}^{m} w_{i2} v_{ij2}, \sum_{i=1}^{m} w_{i3} v_{ij3} \right). \tag{34}$$

A Numerical Example (Bector et al., 2002)

Discount Saving Bonds (DSB). Bector et al. (2002) assumed that the fuzzy ratings and the fuzzy weights for the DSB are given in the form of TFNs in Table 9 along with their $\alpha-$cuts.

Table 9. Computation of Fuzzy Score for Discount Saving Bonds (DSB)

Criterion	Weight (α)	Rating (α)	Score (α)=Weight (α) $\otimes$ Rating (α)
Terms & availability	$(2+\alpha,\ 4-\alpha)$	$(4+\alpha,\ 6-\alpha)$	$(8+6\alpha+\alpha^2,\ 24-10\alpha+\alpha^2)$
Quality of the bonds	$(5+\alpha,\ 7-\alpha)$	$(7+\alpha,\ 9-\alpha)$	$(35+12\alpha+\alpha^2,\ 63-16\alpha+\alpha^2)$
Backed by support	$(6+\alpha,\ 8-\alpha)$	$(8+\alpha,\ 10-\alpha)$	$(48+14\alpha+\alpha^2,\ 80-18\alpha+\alpha^2)$
Liquidity	$(5+\alpha,\ 7-\alpha)$	$(5+\alpha,\ 7-\alpha)$	$(25+10\alpha+\alpha^2,\ 49-14\alpha+\alpha^2)$
Income frequency from the bonds	$(6+\alpha,\ 8-\alpha)$	$(4+\alpha,\ 6-\alpha)$	$(24+10\alpha+\alpha^2,\ 48-14\alpha+\alpha^2)$
Trade denominations	$(1+\alpha,\ 3-\alpha)$	$(6+\alpha,\ 8-\alpha)$	$(6+7\alpha+\alpha^2,\ 24-11\alpha+\alpha^2)$
Taxation	$(3+\alpha,\ 5-\alpha)$	$(4+\alpha,\ 6-\alpha)$	$(12+7\alpha+\alpha^2,\ 30-11\alpha+\alpha^2)$
Other characteristics	$(2+\alpha,\ 4-\alpha)$	$(4+\alpha,\ 6-\alpha)$	$(8+6\alpha+\alpha^2,\ 24-10\alpha+\alpha^2)$
Recommended for investment	$(3+\alpha,\ 5-\alpha)$	$(3+\alpha,\ 5-\alpha)$	$(9+6\alpha+\alpha^2,\ 25-10\alpha+\alpha^2)$
Total fuzzy score (α)			$(175+78\alpha+9\alpha^2,\ 367+114\alpha+9\alpha^2)$

Shown in Table 9, the total fuzzy score (α) of a DSB is a parabolic fuzzy number. This parabolic fuzzy number can be approximately represented with a triangular fuzzy number by taking $\alpha = 0{,}1$, and 0, respectively: (175, 262, 367).

3.1.7 Vanegas and Labib's (2001) Approach

Vanegas and Labib (2001) propose a novel method of operating on fuzzy numbers to obtain a fuzzy weighted average of desirability levels during engineering design evaluation. The method produces overall desirability levels less imprecise and more realistic than those of the conventional fuzzy weighted average (FWA).

The $\alpha-$ cut of the overall desirability of an alternative *D*, calculated through the new FWA for *n* desirability levels represented by the fuzzy numbers $D_1, D_2, \cdots, D_n$, with weights (fuzzy numbers) $W_1, W_2, \cdots, W_n$, is given by

$$D_\alpha = \left[D_{\alpha a}, D_{\alpha b}\right] \tag{35}$$

where

$$D_{\alpha a} = \min\left(\frac{\sum_{i=1}^{n} D_{i\alpha a} \times w_i}{\sum_{i=1}^{n} w_i}\right) \tag{36}$$

and

$$D_{\alpha b} = \max\left(\frac{\sum_{i=1}^{n} D_{i\alpha b} \times w_i}{\sum_{i=1}^{n} w_i}\right) \tag{37}$$

where $w_i \in \left[W_{i\alpha a}, W_{i\alpha b}\right]$, for all $i \in \{1,2,...,n\}$ and all $\alpha \in (0,1]$.

$D_{\alpha a}$ and $D_{\alpha b}$ represent the lower and upper limits, respectively, of the $\alpha-$ cut $D_{i\alpha}$; and $D_{i\alpha a}$ and $D_{i\alpha b}$ represent the lower and upper limits, respectively, of the $\alpha-$ cut $D_{i\alpha}$; and $W_{i\alpha a}$ and $W_{i\alpha b}$ represent the lower and upper limits, respectively, of the $\alpha-$ cut $W_{i\alpha}$. The "min" and "max" operators take the minimum and maximum values, respectively, that can be calculated through the combination of the w_i in all the possible ways. The set of w_i that is used in the numerator has to be the same as the one in the denominator.

3.2 Fuzzy Multiplicative Weighting Method

In the fuzzy case, we can use the fuzzy numbers instead of the crisp ones in Eq. (4). Thus, we have

$$\tilde{V}(A_i) = \left(\prod_{j=1}^{n} x_{lij}^{w_{li}}, \prod_{j=1}^{n} x_{mij}^{w_{mi}}, \prod_{j=1}^{n} x_{uij}^{w_{ui}} \right) \tag{38}$$

where x_{kij}^{wki} is the score or value of the *k*th parameter ($k = l$, m, and u) of the criterion *j* of the alternative *i*, weighted by the fuzzy weight of the same criterion. The ranking of $\tilde{V}(A_i)$s can be made by using any ranking method.

A Numerical Example

Two FMS alternatives will be evaluated using the criteria *engineering effort* (X_1), *flexibility* (X_2), *net present worth* (X_3), and *integration ability* (X_4). The criteria weights and the alternative scores with respect to each alternative are given in Table 10. The results of the problem are illustrated in Figure 6. It is clearly seen that FMS-1 should be selected.

Table 10. Data for the Numerical Example

Criteria	Criteria Weights	FMS-1	FMS-2
engineering effort (X_1)	(0.15, 0.18, 0.24)	(3, 4, 6)	(5, 6, 7)
flexibility (X_2)	(0.32, 0.38, 0.46)	(6, 8, 10)	(4, 4, 5)
net present worth (X_3)	(0.30, 0.32, 0.38)	(28, 32, 44)	(38, 45, 52)
integration ability (X_4)	(0.10, 0.12, 0.18)	(6, 8, 9)	(3, 4, 5)

$$\tilde{V}(FMS_1) = \left(\prod_{j=1}^{4} x_{l1j}^{w_{l1}}, \prod_{j=1}^{4} x_{m1j}^{w_{m1}}, \prod_{j=1}^{4} x_{u1j}^{w_{u1}} \right)$$

$$= \begin{pmatrix} 3^{0.15} \times 6^{0.32} \times 28^{0.30} \times 6^{0.10}, \; 4^{0.18} \times 8^{0.38} \times 32^{0.32} \times 8^{0.12}, \\ 6^{0.24} \times 10^{0.46} \times 44^{0.38} \times 9^{0.18} \end{pmatrix}$$

$$= (6.8005, \; 11.0043\;, \; 27.7352)$$

$$\tilde{V}\left(FMS_2\right)=\left(\prod_{j=1}^{4} x_{l2j}^{w_{l2}}, \prod_{j=1}^{4} x_{m2j}^{w_{m2}}, \prod_{j=1}^{4} x_{u2j}^{w_{u2}}\right)$$

$$=\begin{pmatrix} 5^{0.15}\times 4^{0.32}\times 38^{0.30}\times 3^{0.10},\ 6^{0.18}\times 4^{0.38}\times 45^{0.32}\times 4^{0.12}, \\ 7^{0.24}\times 5^{0.46}\times 52^{0.38}\times 5^{0.18} \end{pmatrix}$$

$$=\left(6.5940,\ 9.3352,\ 20.0560\right)$$

What would your selection be if you had used the fuzzy SAW method (Bector et al., 2002) instead of fuzzy multiplicative weighting method? If you had, you would obtain the following results which indicate that FMS_2 should be selected this time:

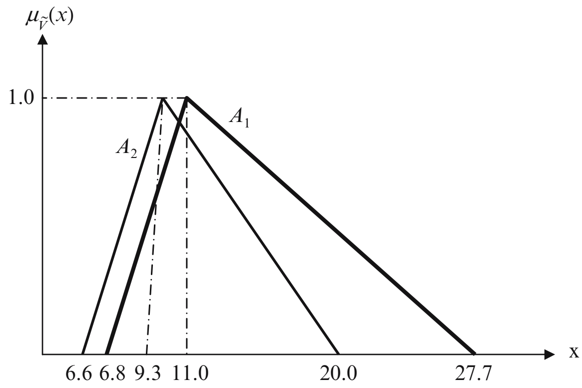

Figure 6. Fuzzy multiplicative scores of FMS alternatives

$$\tilde{V}\left(\text{FMS}_1\right)=\left(11.37, 14.96, 24.38\right)$$

$$\tilde{V}\left(\text{FMS}_2\right)=\left(13.73, 17.48, 24.64\right)$$

The reason for this change in the ranking is that the multiplicative model gives a smaller value with x^w, $0 \prec w \prec 1$, than the result with $x \times w, 0 \prec w \prec 1$.

4. CONCLUSIONS

According to the data type of the alternative's performance, fuzzy multi-attribute decision-making methods can be categorized into three groups: (1) data are all fuzzy, (2) all crisp, and (3) either crisp or fuzzy. The methods in the third group are either too cumbersome to use or only suitable for the purpose of screening out unsuitable alternatives. The fuzzy MADM methods with data type is all fuzzy require transforming crisp data to fuzzy numbers, despite that the data are crisp in nature, which not only violates the intention of fuzzy set theory, but also increases the decision complexity.

The fuzzy weighted scoring models are widely used in the literature. Simple Additive Weighting Method is probably the best known and widely used method. The overall score of an alternative is computed as the weighted sum of all the attribute values. It is simple and easy to understand. Multiplicative weighting methods are superior to SAW methods because they do not need the data to be normalized. The data with different units can be directly used in multiplicative methods.

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FUZZY MULTI-ATTRIBUTE DECISION MAKING USING AN INFORMATION AXIOM-BASED APPROACH

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Abstract: Axiomatic design (AD) provides a framework to describe design objects and a set of axioms to evaluate relations between intended functions and the means by which they are achieved. Since AD has the characteristics of multi-attribute evaluation, it is proposed for multi-attribute comparison of some alternatives. The comparison of these alternatives is made for the cases of both complete and incomplete information. The crisp AD approach for complete information and the fuzzy AD approach for incomplete information are developed. In this chapter, the numeric applications of both crisp and fuzzy AD approaches for the comparison of flexible-manufacturing systems are given.

Key words: Axiomatic design, multi-attribute, information axiom, flexible manufacturing

1. INTRODUCTION

Approaches that include more than one measure of performance in the evaluation process are termed multi-attribute or multi-criteria decision methods. The advantage of these methods is that they can account for both financial and nonfinancial impacts. Among these methods, the most popular ones are scoring models (Nelson, 1986), analytic hierarchy process (AHP) (Kahraman et al., 2004), analytic network process (ANP) (Büyüközkan et al., 2004), utility models (Suh, 1995), order preference by similarity ideal solution (TOPSIS) (Deng et al., 2000), and outranking methods (De Boer et al., 1998). Axiomatic design principles, including the information axiom also, present an opportunity for multi-attribute evaluation.

The axiomatic design process is described by the mapping process from functional requirements (FRs) to design parameters (DPs). The goal in axiomatic design is to satisfy the goals of the customer domain through accomplishment in the subsequent domains, which requires mapping from one space to the next. In the mapping (design) process, Suh (1990) imposes two axioms that must be followed in order to create the "best" design. The information axiom (IA), which is the second axiom of AD, proposes the selection of the proper alternative that has minimum information content.

Having to use crisp values is one of the problematic points in the crisp evaluation process. As some criteria are difficult to measure by crisp values, they are usually neglected during the evaluation. Another reason is about mathematical models that are based on crisp values. These methods cannot deal with decision makers' ambiguities, uncertainties, and vagueness, which cannot be handled by crisp values. The use of fuzzy set theory (Zadeh, 1965) allows the decision makers to incorporate unquantifiable information, incomplete information, nonobtainable information, and partially ignorant facts into the decision model.

A model based on IA enables decision makers to evaluate both qualitative and quantitative criteria together. In this chapter, a crisp multi-attribute information axiom (IA) approach and then a fuzzy multi-attribute IA approach for multi-attribute decision-making problems will be developed and the implementation process will be shown by the real-world examples.

2. PRINCIPLES OF AXIOMATIC DESIGN

The most important concept in axiomatic design is the existence of the design axioms. The first design axiom is known as the independence axiom, and the second axiom is known as the information axiom. They are stated as follows (Suh, 1990).

Axiom 1. The Independence Axiom: Maintain the independence of functional requirements.

Axiom 2. The Information Axiom: Minimize the information content.

The independence axiom states that the independence of FRs must always be maintained where FRs are defined as the minimum set of independent requirements that characterize the design goals. The information axiom states that the design with the smallest information content among those satisfying the first axiom is the best design (Suh, 2001).

2.1 Crisp Information Axiom

Information is defined in terms of the information content, I, that is related in its simplest form to the probability of satisfying the given FRs. Information content I_i for a given FR_i is defined as follows:

$$\mathrm{I}_i = \log_2 \left(\frac{1}{p_i} \right) \tag{1}$$

where p_i is the probability of achieving the functional requirement FR_i and log is the logarithm in base 2 (with the unit of bits). This definition of information follows the definition of Shannon (1948), although there are operational differences. Because there are n FRs, the total information content is the sum of all these probabilities. If I_i approaches infinity, the system will never work. When all probabilities are one, the information content is zero, and conversely, the information required is infinite when one or more probabilities are equal to zero (Suh, 1995).

In any design situation, the probability of success is given by what the designer wishes to achieve in terms of tolerance (i.e., design range) and what the system is capable of delivering (i.e., system range). As shown in Figure 1, the overlap between the designer-specified "design range" and the system capability range "system range" is the region where the acceptable solution exists. Therefore, in the case of a uniform probability distribution function, p_i may be written as

$$p_i = \left(\frac{\text{Common range}}{\text{System range}} \right). \tag{2}$$

Therefore, the information content is equal to

$$I_i = \log_2 \left(\frac{\text{System range}}{\text{Common range}} \right). \tag{3}$$

The probability of achieving FR_i in the design range may be expressed, if FR_i is a continuous random variable, as

$$p_i = \int_{dr^1}^{dr^u} p_s(\text{FR}).d\text{FR} \tag{4}$$

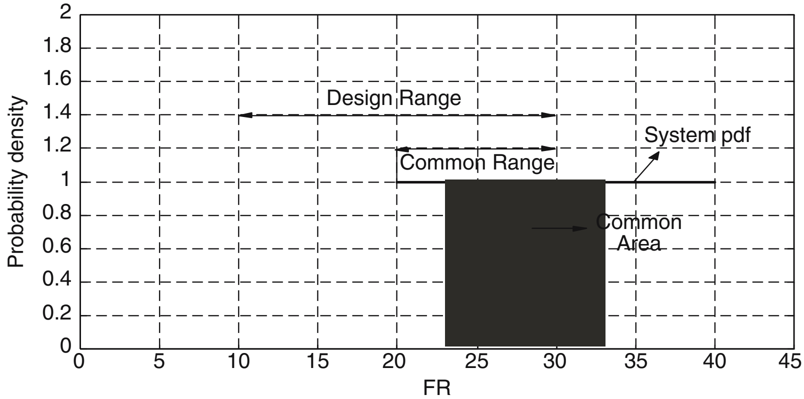

Figure 1. Design range, system range, common range, and probability density function (pdf) of an FR

where p_s (FR) is the system pdf (probability density function) for FR. Eq. (4) gives the probability of success by integrating the system pdf over the entire design range. (i.e., the lower bound of design range, dr^1, to the upper bound of the design range, dr^u). In Figure 2, the area of the common range (A_{cr}) is equal to the probability of success P (Suh, 1990).

Therefore, the information content is equal to

$$I = \log_2\left(\frac{1}{A_{cr}}\right). \tag{5}$$

The information content in Eq. (1) is a kind of entropy that measures uncertainty. There are some other measures of information in terms of uncertainty. Prior to the theory of fuzzy sets, two principal measures of uncertainty were recognized. One of them, proposed by Hartley (1928), is based solely on the classic set theory. The other, introduced by Shannon (1948), is formulated in terms of probability theory. Both of these measures pertain to some aspects of ambiguity, as opposed to vagueness or fuzziness. Both Hartley and Shannon introduced their measures for the purpose of measuring information in terms of uncertainty. Therefore, these measures are often referred to as measures of information. The measure invented by Shannon is referred to as the Shannon entropy.

The Shannon entropy, which is a measure of uncertainty and information formulated in terms of probability theory, is expressed by the function

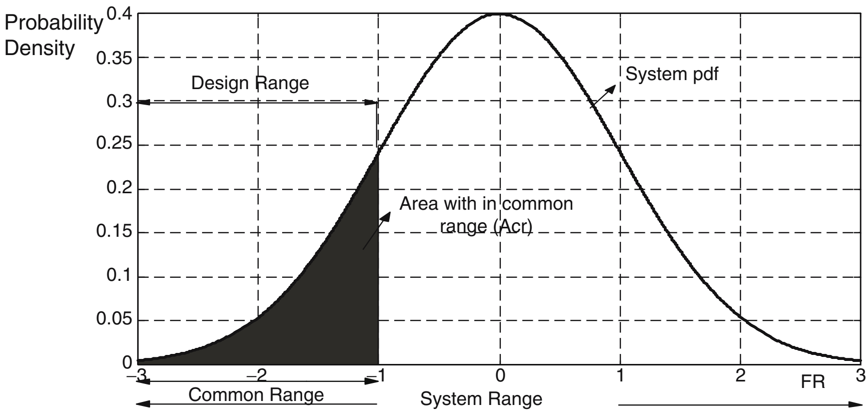

Figure 2. Design range, system range, common range, and pdf of a FR

$$H\left(p(x)/\, x \in X\right) = -\sum_{x \in X} p(x) \log_2 p(x) \tag{6}$$

where $\left(p(x)/\, x \in X\right)$ is a probability distribution on a finite set X.

Suh's entropy in axiomatic design does not require that the total of the probabilities be equal to 1.0, whereas Shannon entropy does. Because of this property, Shannon entropy should not be used as an entropy measure while evaluating independent functional requirements in axiomatic design.

2.2 Fuzzy Information Axiom Approach

The multi-attribute crisp information axiom approach mentioned before can be used for the solution of decision-making problems under certainty. This approach cannot be used with incomplete information, since the expression of decision variables by crisp numbers would be ill defined. For this reason, the multi-attribute fuzzy information axiom is developed in this study. At the same time, a problem including both crisp and fuzzy criteria can be solved by integrating crisp and fuzzy information axiom approaches. This feature is an important advantage that can not be found in other multi-attribute approaches. The definition and formulation of the developed fuzzy approach are given in the following discussion.

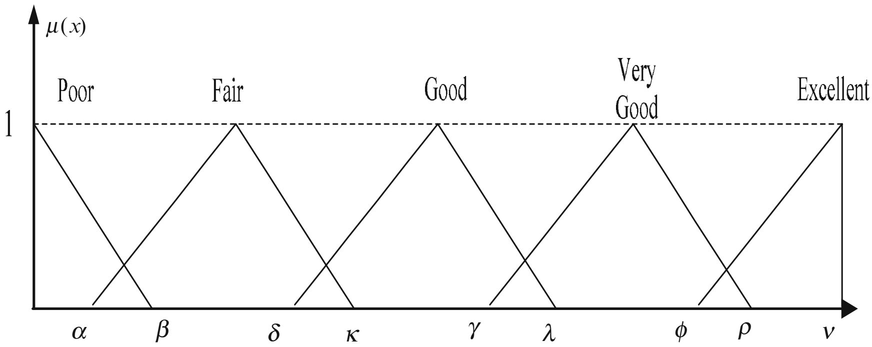

Figure 3. The numerical approximation system for intangible factors

The data relevant to the criteria under incomplete information can be expressed as fuzzy data. The fuzzy data can be linguistic terms, fuzzy sets, or fuzzy numbers. If the fuzzy data are linguistic terms, they are transformed into fuzzy numbers first. Then all the fuzzy numbers (or fuzzy sets) are assigned crisp scores. The following numerical approximation systems are proposed to systematically convert linguistic terms into their corresponding fuzzy numbers. The system contains five conversion scales (Figures 3 and 4).

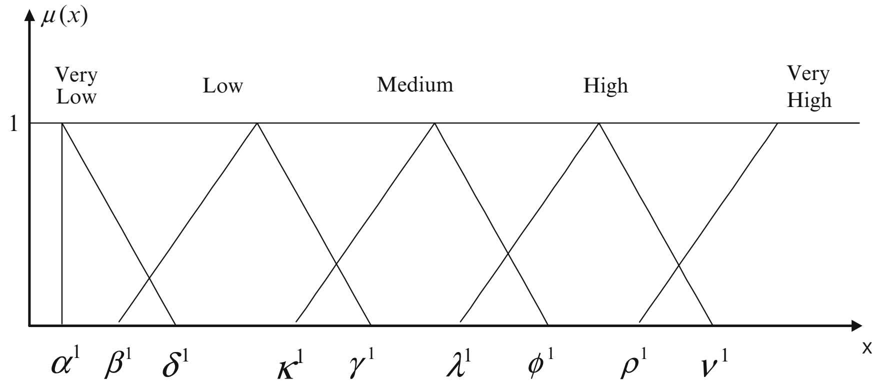

Figure 4. The numerical approximation system for tangible factors

In the fuzzy case, we have incomplete information about the system and design ranges. The system and design range for a certain criterion will be expressed by using "over a number," "around a number," or "between two numbers." Triangular or trapezoidal fuzzy numbers can represent these kinds of expressions. We now have a membership function of a triangular or trapezoidal fuzzy number, whereas we have a probability density function in the crisp case. So, the common area is the intersection area of triangular or trapezoidal fuzzy numbers. The common area between design range and system range is shown in Figure 5.

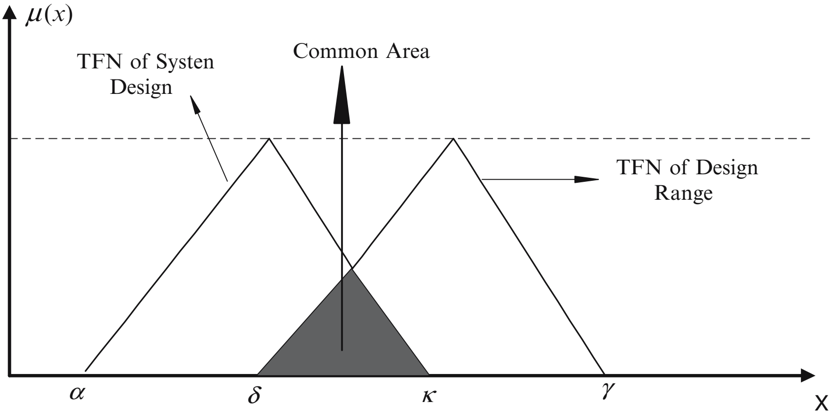

Figure 5. The common area of system and design ranges

Therefore, information content is equal to

$$I = \log_2\left(\frac{\text{TFN of System Design}}{\text{Common Area}}\right). \tag{7}$$

In the following section, the numerical application of these approaches for solving multi-attribute decision-making problems is given.

3. MULTI-ATTRIBUTE COMPARISON OF ADVANCED MANUFACTURING SYSTEMS

The term "advanced manufacturing systems" (AMS) is broadly defined to include any automated (usually computer oriented) technology used in design, manufacturing/service, and decision support. Components of AMS include computer-aided engineering, factory management and control systems, computer-integrated manufacturing processes, and information integration. Many factories reach an intermediate stage, often-called flexible manufacturing systems (FMS). At this stage some machine tools, material-handling equipment, and other programmable devices are under the integrated control of a computer. FMS can manufacture a wide range of products in batch sizes from one to thousands. They provide many important benefits such as greater manufacturing flexibility, reduced inventory, reduced floor space, faster response to shifts in market demand, lower lead times, and a longer useful life of equipment over successive generations of products. Like many real-world problems, the decision of investing in advanced manufacturing technology frequently involves multiple and conflicting objectives, e.g., minimizing costs, maximizing flexibility, minimizing machine downtimes, or maximizing efficiency (Kulak and Kahraman, 2005).

3.1 A Numerical Application of Crisp Information Axiom

A company manufacturing tractor components wants to renew the manufacturing system. In order to produce a group of products, the company must decide to select the most appropriate one among the different alternative flexible manufacturing systems. With respect to the characteristics of the product group manufactured by a company, the functional requirements that should be satisfied by a flexible manufacturing system are given below. Since FR_1 is a monetary criterion, it is a different criterion from the others. The other criteria are graded between 1 and 20. This grading is arranged to show that the interval 17−20 is excellent, 13−16 is very good, 9−12 is good, 5−8 is fair, and 1−4 is poor.

FR_1= Annual Depreciation and Maintenance Cost (*ADMC*) must be in the range of $100,000 to $200,000,
FR_2= Quality of Results (QR) must be over 9,
FR_3= Ease of use (EU) must be over 13,
FR_4= Competitive (C) must be in the range of 15 to 18,
FR_5= Adaptability (A) must be over 15,
FR_6= Expandability (E) must be in the range of 12 to 16.

Alternative flexible-manufacturing systems' annual depreciation and maintenance costs and performance scores evaluated by the experts with respect to certain criteria are given in Table 1. The data given in the Table 1 are arranged to include the minimum and maximum performance values supplied by the system.

Table 1. The System Range Data for Advanced Manufacturing Systems

AMSs	ADMC (*$1000)	QR	EU	C	A	E
FMS-I	210 to 240	18 to 20	13 to 18	16 to 20	12 to 18	12 to 16
FMS-II	80 to 120	12 to 17	9 to 14	12 to 17	15 to 17	14 to 18
FMS-III	180 to 220	8 to 12	10 to 14	13 to 18	19 to 20	9 to 14
FMS-IV	140 to 170	7 to 10	8 to 14	13 to 17	12 to 16	11 to 13

The data in Table 1, related to annual depreciation and maintenance costs, reflect only the minimum and maximum cost values. The ADMC costs of the alternatives in Table 1 have the probability density functions as shown in Table 2.

Table 2. The Probability Density Functions of ADMC

AMS	The Probability Density Functions of ADMC	Range ($100,000)
FMS-I	$f(x) = 0.697x^2 - 0.2$	$2.1 \leq x \leq 2.4$
FMS-II	$f(x) = 2.404x^3$	$0.8 \leq x \leq 1.2$
FMS-III	$f(x) = 0.723x^2 - 0.4$	$1.8 \leq x \leq 2.2$
FMS-IV	$f(x) = 1.591x^2 - 0.5$	$1.4 \leq x \leq 1.7$

Using these design and system ranges, the information content for each FR in each FMS may be computed using Equations (3) and (5). Some sample calculations to obtain the information contents of *ADMC* and *QR* are presented below.

Annual Depreciation and Maintenance Cost

For FMS-I:

$A_{cr} = 0$

$$I_{\text{ADMC}-1} = \log_2\left(\frac{1}{A_{cr}}\right) = \infty \tag{8}$$

For FMS-II:

$$A_{cr} = \int_{1}^{1.2} 2.404x^3 dx = 0.645 \quad \mathrm{I}_{\mathrm{ADMC-2}} = \log_2\left(\frac{1}{\mathrm{A}_{\mathrm{cr}}}\right) = 0.633 \qquad (9)$$

The design and system ranges of ADMC for FMS-I and FMS-II are shown in Figures 6 and 7, respectively. And the design and systems ranges of QR for FMS are also shown in Figures 8–13.

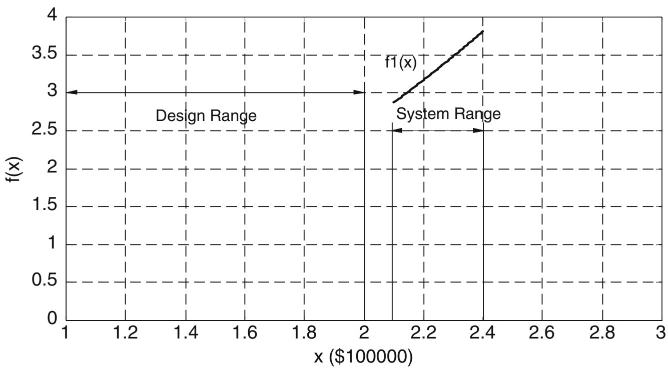

Figure 6. Design and system ranges of ADMC for FMS-I

For FMS-III:

$$A_{cr} = \int_{1.8}^{2}\left(0.723x^2 - \frac{2}{5}\right)dx = 0.442 \quad I_{\mathrm{ADMC-3}} = \log_2\left(\frac{1}{A_{cr}}\right) = 1.178 \qquad (10)$$

For FMS-IV:

$$A_{cr} = \int_{1.4}^{1.7}\left(1.591x^2 - \frac{1}{2}\right)dx = 1 \quad I_{\mathrm{ADMC-4}} = \log_2\left(\frac{1}{A_{cr}}\right) = 0 \qquad (11)$$

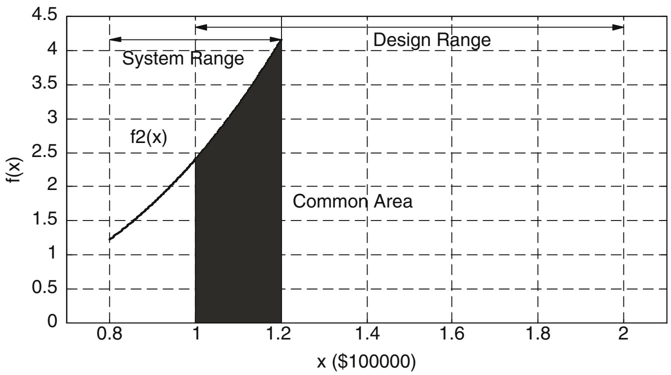

Figure 7. Design and system ranges of ADMC for FMS-II

Quality of Results

For FMS-I:

$$I_{\mathrm{QR}-1} = \log_2\left(\frac{20-18}{20-18}\right) = \log_2(1) = 0 \tag{12}$$

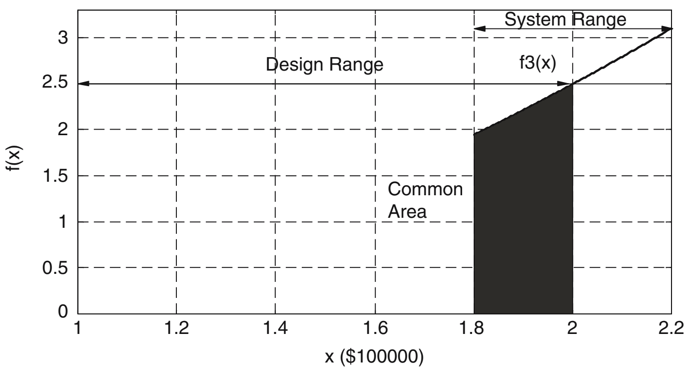

Figure 8. Design and system ranges of ADMC for FMS-III

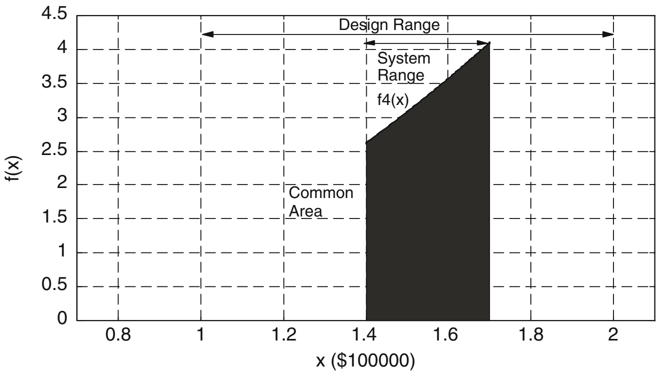

Figure 9. Design and system ranges of ADMC for FMS-IV

For FMS-II:

$$I_{\mathrm{QR}-2} = \log_2\left(\frac{17-12}{17-12}\right) = \log_2(1) = 0 \qquad (13)$$

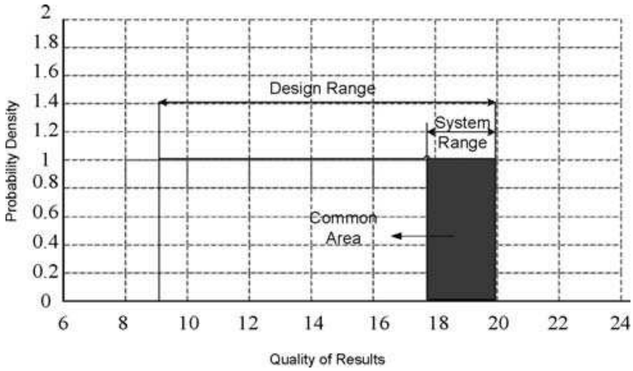

Figure 10. Design and system ranges of QR for FMS-I

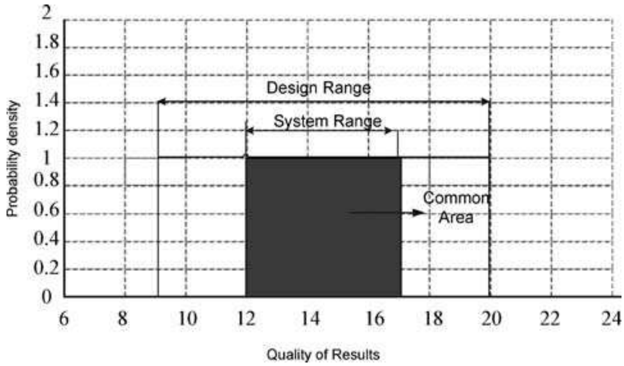

Figure 11. Design and system ranges of QR for FMS-II

For FMS-III:

$$I_{\mathrm{QR}-3} = \log_2\left(\frac{12-8}{12-9}\right) = \log_2\left(\frac{4}{3}\right) = 0,415 \tag{14}$$

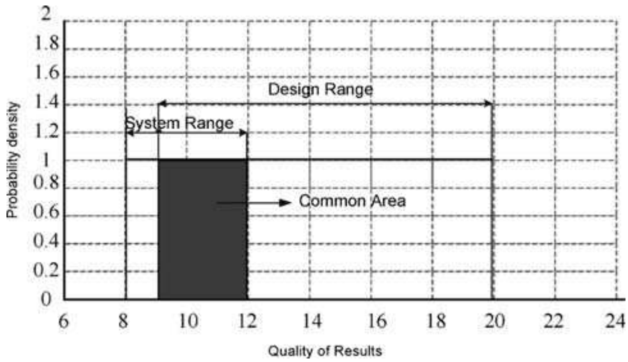

Figure 12. Design and system ranges of QR for FMS-III

For FMS-IV:

$$I_{\mathrm{QR}-4} = \log_2\left(\frac{10-7}{10-9}\right) = \log_2(3) = 1,585 \tag{15}$$

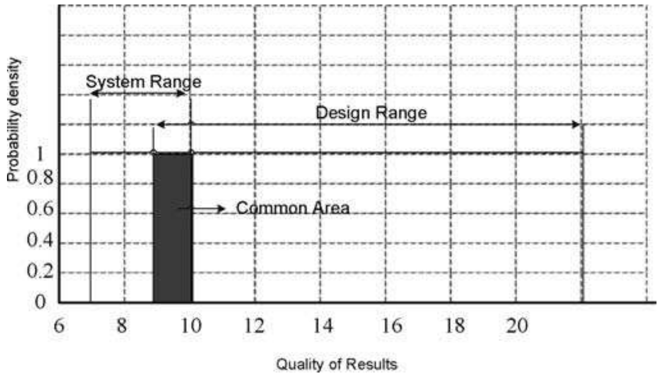

Figure 13. Design and system ranges of QR for FMS-IV

The information contents for the other criteria with respect to the alternatives are given in Table 3. As the system with minimum information content is the best one, FMS-II is selected.

Table 3. Suh's Information Content for Advanced Manufacturing Systems

AMSs	I_{ADMC}	I_{QR}	I_{EU}	I_C	I_A	I_E	ΣI_i
FMS-I	Infinite	0.000	0.000	0.415	1.000	0.000	Infinite
FMS-II	0.633	0.000	2.322	1.322	0.000	1.000	5.277*
FMS-III	1.178	0.415	2.000	0.737	0.000	1.322	5.652
FMS-IV	0.000	1.585	2.585	1.000	2.000	1.000	8.170

3.2 A Numerical Application of Fuzzy Information Axiom

The same company in Section 3.1 has the following fuzzy functional requirements:

FR_1 = ADMC must be low,
FR_2 = QR must be very good,
FR_3 = EU must be very good,
FR_4 = C must be excellent,
FR_5 = A must be excellent,
FR_6 = E must be very good.

The experts produce the system range data and use the linguistic expressions as in Table 4.

Table 4. The System Range Data for Advanced Manufacturing Systems

AMS	ADMC	QR	EU	C	A	E
FMS-I	High	Excellent	Very good	Excellent	Very good	Very good
FMS-II	Very Low	Very good	Good	Very good	Very good	Very good
FMS-III	Medium	Good	Good	Very good	Excellent	Good
FMS-IV	Low	Fair	Good	Very good	Very good	Good

The conversation scales for intangibles are given in Figure 14 whereas the ones for ADMC are given in Figure 15.

In order to obtain the information content for ADMC and QR two sample calculations are given in the following.

Annual Depreciation and Maintenance Cost

For FMS-III:
Common Area = (180 − 170) × 0.2 / 2 = 1
System Area = (210 − 170) × 1 / 2 = 20

$$I_{\mathrm{ADMC}} = \log_2\left(\frac{\text{System Area}}{\text{Common Area}}\right) = \mathrm{Log}_2\left(\frac{20}{1}\right) = 4.322 \qquad (16)$$

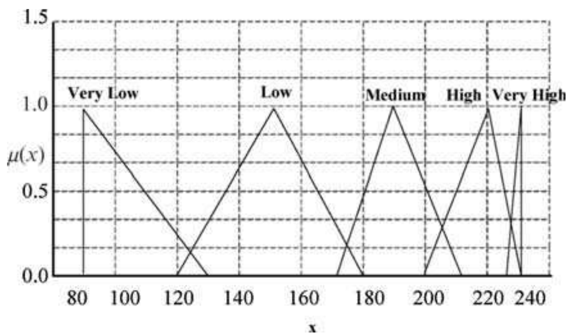

Figure 14. TFNs for intangible factors

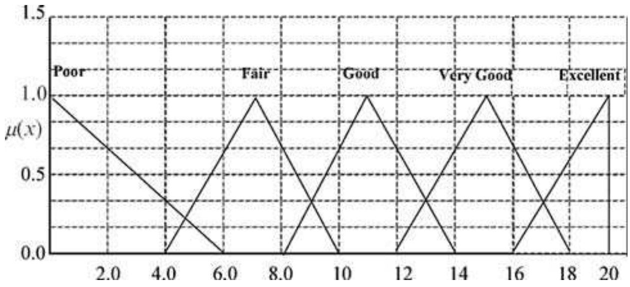

Figure 15. TFNs for tangible factors

Quality of results

For FMS-III:
Common Area = (14 − 12) × 0.333 / 2 = 0.333
System Area = (14 − 8) × 1 / 2 = 3

$$I_{QR} = \log_2\left(\frac{\text{System Area}}{\text{Common Area}}\right) = \log_2\left(\frac{3}{0.333}\right) = 3.171 \quad (17)$$

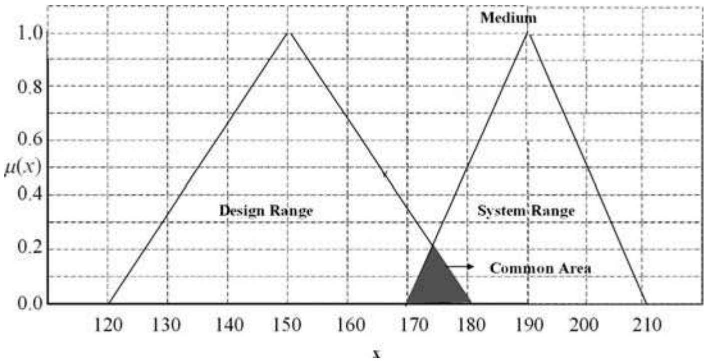

Figure 16. Design and system ranges of ADMC in case of fuzziness

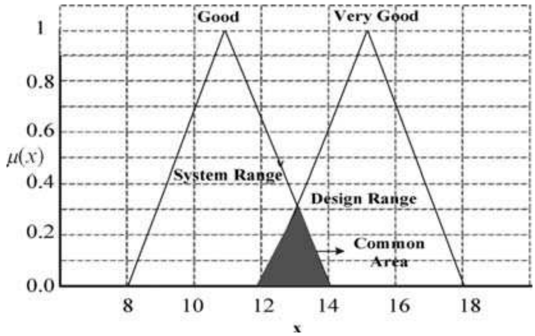

Figure 17. Design and system ranges of QR in case of fuzziness

The information contents for the other criteria with respect to the alternatives are given in Table 5. The alternative with minimum information content is FMS-II.

Table 5. The information content for advanced manufacturing systems

AMSs	I_{ADMC}	I_{QR}	I_{EU}	I_C	I_A	I_E	ΣI_i
FMS-I	Infinite	2.806	0.000	0.000	3.391	0.000	Infinite
FMS-II	5.322	0.000	3.171	3.391	3.391	0.001	15.275*
FMS-III	4.322	3.171	3.171	3.391	0.000	3.171	17.000
FMS-IV	0.000	Infinite	3.171	3.391	3.391	3.171	Infinite

The rankings obtained by using the crisp and fuzzy approaches are the same. When the attribute ADMC is excluded in the evaluation above, FMS-I will be the best alternative. Although FMS-I is the best alternative having the minimum information content in total for all the criteria except ADMC, FMS-I is not selected since the ADMC system and design ranges are not overlapped.

4. MULTI-ATTRIBUTE EQUIPMENT SELECTION

The satisfaction of customer requirements forces companies to become more sensitive and to make deep analyses in selecting equipment. The selection of oversized equipment can disturb the company's cash flow and also the problems such as excessive inventory and idle equipment can be met. On the contrary, the selection of under-sizing equipment cannot fulfill requested quality levels and capacity requirements by customers.

Equipment selection is also an important decision-making problem for the design of a flexible manufacturing system (Kulak et al., 2005)

An international company needs a few punching machines to manufacture its products, racks, and sub-racks in which electronic materials are located. The company determined six possible punching machines with respect to the manufacturing requirements. The criteria considered in the selection process are categorized into the groups of costs and technical characteristics. The group of costs includes fixed costs per hour, variable costs per hour, and equivalent costs of standard tools per hour. The group of technical characteristics includes length of sheet size, thickness of sheet metal, number of strokes for 25-mm pitchsize sheet metal, simultaneous axis speed, tool rotation speed, and sufficiency of service. Some criteria including *positioning the work piece precisely* and *width of sheet metal* are excluded since the values of these criteria are the same for each candidate.

The criteria in the group of costs are linguistic variables. The sufficiency of service in the group of technical characteristics is also a linguistic variable. The company's design ranges, which means that what a designer wants to achieve for the above criteria are as follows:

FR_{FC} = Fixed costs per hour *(FC)* must be medium,
FR_{VC} = Variable costs per hour *(VC)* must be low,
FR_{ST} = Equivalent costs of standard tools per hour *(ST)* must be low,
FR_{L} = Length of sheet size *(L)* must be in the range of 1200 to 2540,
FR_{T} = Thickness of sheet metal *(T)* must be in the range of 3 to 8,
FR_{NS} = Number of strokes for 25 mm pitchsize sheet metal *(NS)* must be in the range of 190 to 445,
FR_{XY} = Simultaneous axis speed *(XY)* must be in the range of 70 to 110,
FR_{SR} = Tool rotation speed *(SR)* must be in the range of 50 to 180,
FR_{SS} = Sufficiency of service *(SS)* must be excellent.

Alternative punching machines' costs and performance scores evaluated by the company's managers with respect to criteria are given in Tables 6 and 7. The data for design ranges and the data for system ranges are entered into the software-MAXD. The calculated results below are obtained by MAXD. The data given in Table 7, except sufficiency of service, are arranged to include the minimum and maximum performance values supplied by the punching machines. The managers produce the system range data and use the linguistic expressions about costs and sufficiency of service as in Tables 6 and 7.

Figures 18, 19, and 20 show the membership functions of the linguistic expressions about fixed costs per hour, variable costs per hour, and equivalent costs of standard tools per hour, respectively. Figure 21 also shows the membership functions of the linguistic expressions about sufficiency of service. For example, in Figure 18, the decision maker subjectively evaluates the alternatives with the linguistic term "*very low*" if it is assigned a score of (8, 8, 10); "*low*" with a score of (8, 10, 12); "*medium*" with a score of (10, 12, 14); "*high*" with a score of (12, 14, 16); and "*very high*" with a score of (14, 16, 16).

Table 6. The System Range Data for Costs

Alternative	Criteria		
Punch Equipments	Fixed costs per hour (Euro/hour)	Variable costs per hour (Euro/hour)	Equivalent costs of standard tools per hour (Euro/hour)
Punch-A	Low	Very Low	Low
Punch-B	Medium	Medium	Low
Punch-C	High	Medium	Low
Punch-D	Very high	High	Low
Punch-E	Medium	Medium	High
Punch-F	Low	Low	Medium

Table 7. The System Range Data for Technical Characteristics

Alternative	Criteria					
Punching Machines	Length of sheet size (mm)	Thickness of sheet metal (mm)	Number of strokes for 25−mm pitchsize sheet metal	Simultaneous axis speed (m/min.)	Tool rotation speed (rpm)	Sufficiency of service
Punch-A	0 to 1270	0 to 6,4	0 to 420	0 to 108	0 to 180	Excellent
Punch-B	0 to 2070	0 to 6,4	0 to 220	0 to 97	0 to 60	Excellent
Punch-C	0 to 2540	0 to 6,4	0 to 445	0 to 108	0 to 180	Excellent
Punch-D	0 to 2535	0 to 8,0	0 to 445	0 to 108	0 to 60	Excellent
Punch-E	0 to 2500	0 to 6,4	0 to 400	0 to 110	0 to 60	Very Good
Punch-F	0 to 1270	0 to 6,4	0 to 200	0 to 82	0 to 60	Very Good

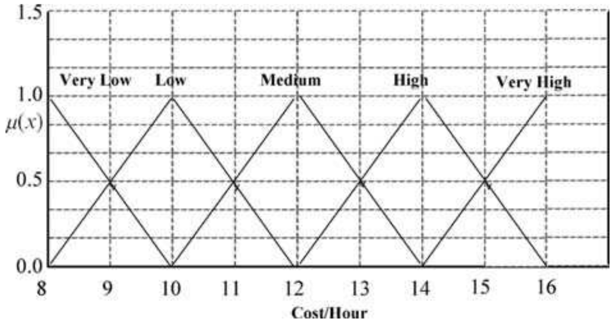

Figure 18. TFNs for tangible factors (fixed costs)

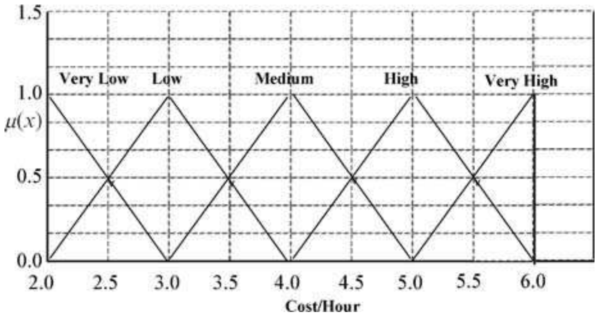

Figure 19. TFNs for tangible factors (variable costs)

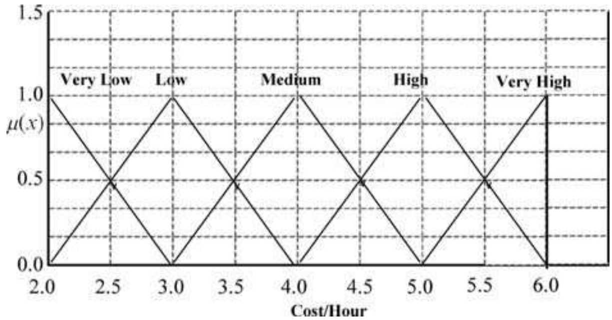

Figure 20. TFNs for tangible factors (equivalent costs of standard tools per hour)

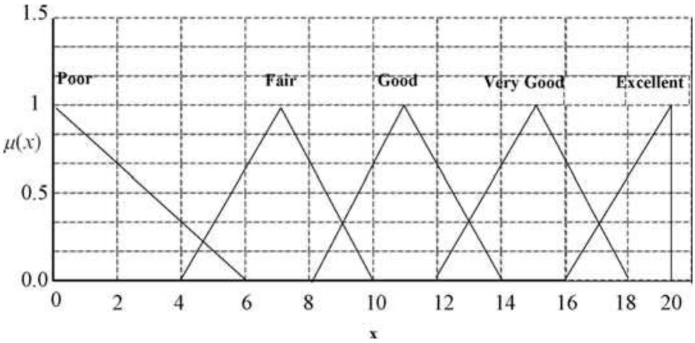

Figure 21. TFNs for intangible factors (sufficiency of service)

In the following section, unweighted and weighted multi-attribute IA approaches will be applied to the equipment selection problem above.

4.1 Unweighted Multi-Attribute IA Approach

The information content for Punch-B can be computed using Eq. (3) with the system range in Table 7 and the design range for the length of metal-sheet (L) above.

For Punch-B :

Common Area = (2070 − 1200) × 1 = 870

System Area= (2070 − 0) × 1 = 2070

$$I_L = \log_2\left(\frac{\text{System Area}}{\text{Common Area}}\right) = \log_2\left(\frac{2070}{870}\right) = 1.250 \qquad (18)$$

The information content for Punch-A can be computed using Eq. (7) with the system range in Table 6 and the design range for the fixed costs (FC).

For Punch-A :

Common Area= (12 − 10) × 0.5 / 2 = 0.5

System Area= (12 − 8) × 1 / 2 = 2

$$I_{FC} = \log_2\left(\frac{\text{System Area}}{\text{Common Area}}\right) = \log_2\left(\frac{2}{0.5}\right) = 2.000 \qquad (19)$$

For this approach, the results in Table 8 are obtained in a similar way that the sample numerical results are calculated.

Table 8. The Results of Suh's Information Content for Punching Machines

Punching Machines	I_{FC}	I_{VC}	I_{ST}	I_L	I_T	I_{NS}	I_{XY}	I_{SR}	I_{SS}	$\sum I$
A	2.000	1.000	0.000	4.181	0.912	0.869	1.507	0.470	0.000	10.939
B	0.000	2.000	0.000	1.250	0.912	2.874	1.845	2.585	0.000	11.467
C	2.000	2.000	0.000	0.923	0.912	0.803	1.507	0.470	0.000	8.615*
D	Infinite	Infinite	0.000	0.925	0.678	0.803	1.507	2.585	0.000	Infinite
E	0.000	2.000	Infinite	0.943	0.912	0.930	1.459	2.585	3.391	Infinite
F	2.000	0.000	2.000	4.181	0.912	4.322	2.773	2.585	3.391	22.164

The information contents for the criteria with respect to the alternatives are given in Table 8. As the punching machine with minimum information content is the most suitable alternative with respect to the designer's requirements, Punch-C is selected.

In Table 9, the information contents for costs and technical characteristics are given separately since a decision maker may require seeing the effect of any main criterion (costs or technical characteristics). Punch-B is the most suitable alternative with respect to the designer's requirements when the main criteria *costs are* only taken into account. Punch-C is the most suitable alternative with respect to the designer's requirements when the main criteria *technical characteristics* are only taken into account.

Table 9. The Results of Suh's Information Content For Costs And Technical Characteristics

Punch.Mach.	Costs						Technical Characteristics				
	I_{FC}	I_{VC}	I_{ST}	I_L	I_T	$\sum I$	I_{NS}	I_{XY}	I_{SR}	I_{SS}	$\sum I$
A	2.000	1.000	0.000	3.000	4.181	0.912	0.869	1.507	0.470	0.000	7.939
B	0.000	2.000	0.000	2.000*	1.250	0.912	2.874	1.845	2.585	0.000	9.467
C	2.000	2.000	0.000	4.000	0.923	0.912	0.803	1.507	0.470	0.000	4.615*
D	Infinite	Infinite	0.000	Infinite	0.925	0.678	0.803	1.507	2.585	0.000	6.498
E	0.000	2.000	Infinite	Infinite	0.943	0.912	0.930	1.459	2.585	3.391	10.221
F	2.000	0.000	2.000	4.000	4.181	0.912	4.322	2.773	2.585	3.391	18.164

Since each main criterion involves different numbers of subcriteria, the effect of each main criterion on the sum of information contents in Table 8 will possibly be different. In order to remove this effect, the decision maker may use unit indexes for unweighted information content given in Table 10, which are calculated by dividing the total information contents in Table 9 by the number of subcriteria of each main criterion. The column of total unit index in Table 10 is calculated by summing the unit indexes for costs and technical characteristics.

Table 10. Unit Indexes for Unweighted Information Contents

Punching Machines	Index for Costs	Index for Tech. Characteristics	Total Unit Index
A	1.000	1.323	2.323
B	0.667	1.578	2.245
C	1.333	0.769	2.103*
D	Infinite	1.083	Infinite
E	Infinite	1.703	Infinite
F	1.333	3.027	4.361

With respect to the total unit indexes, Punch-C is the selected alternative. Although the same alternative is selected in Table 8 and in Table 10, different alternatives might have been selected. The effect of this approach will be seen in the following section when the weighted multi-attribute IA approach is used.

4.2 Weighted Multi-Attribute IA Approach

In the method in subsection 4.1., the weights for all subcriteria are equal. If the decision maker wants to assign a different weight for each criterion, the following weighted multi-attribute IA approach can be used.

Eq. 20 is proposed for the weighted multi-attribute IA approach:

$$I_{ij} = \begin{cases} \left[\log_2\left(\dfrac{1}{p_{ij}}\right)\right]^{\frac{1}{w_j}}, & 0 \le I_{ij} < 1 \\ \left[\log_2\left(\dfrac{1}{p_{ij}}\right)\right]^{w_j}, & I_{ij} > 1 \\ w_j, & I_{ij} = 1 \end{cases} \tag{20}$$

For this approach, the results in Table 11 are obtained by applying Eq. 20 to the data in Table 8. The weights for the main criteria costs and technical requirements are determined as 0.80 and 0.20, respectively, since the customers of this company give higher importance to the product prices than to the technical characteristics.

Table 11. The Weighted Results for Cost and Technical Characteristics

	Costs						Technical Characteristics				
Punch Mach.	I_{FC}	I_{VC}	I_{ST}	I_L	I_T	$\sum I$	I_{NS}	I_{XY}	I_{SR}	I_{SS}	$\sum I$
A	1.741	1.000	0.000	2.741	1.331	0.632	0.495	1.085	0.023	0.000	3.566
B	0.000	1.741	0.000	1.741	1.046	0.632	1.235	1.130	1.209	0.000	5.252
C	1.741	1.741	0.000	3.482	0.670	0.632	0.334	1.085	0.023	0.000	2.744
D	Infinite	Infinite	0.000	Infinite	0.678	0.143	0.334	1.085	1.209	0.000	3.450
E	0.000	1.741	Infinite	Infinite	0.747	0.632	0.695	1.079	1.209	1.277	5.638
F	1.741	0.000	1.741	3.482	1.331	0.632	1.340	1.226	1.209	1.277	7.015

Punch-B is selected when unit indexes for weighted information contents in Table 11 are used. Table 12 gives unit indexes for weighted information contents. The ranking order when the unweighted approach is used changes as in Table 12 in favor of Punch-B, since the weighted approach reflects the high importance of *costs* to the results.

Table 12. Unit Indexes for Weighted Information Contents

Punching Machines	Index for Costs	Index for Tech. Characteristics	Total Unit Index
A	0.914	0.594	1.508
B	0.580	0.875	1.456*
C	1.161	0.457	1.618
D	Infinite	0.575	Infinite
E	Infinite	0.940	Infinite
F	1.161	1.169	2.330

5. CONCLUSION

Crisp multi-attribute decision making (MADM) methods solve problems in which all decision data are assumed to be known and must be represented by crisp numbers. The methods are to effectively aggregate performance scores. Fuzzy MADM methods have difficulty in judging the preferred alternatives because all aggregated scores are fuzzy data. We propose a crisp multi-attribute IA approach when all decision data are known, whereas we propose fuzzy multi-attribute IA approach when unquantifiable or incomplete information exists. The proposed crisp and fuzzy IA approaches use the design ranges determined by the decision makers to select the best alternative. However, these approaches that depend on the minimum information axiom do not let an alternative be selected even if that alternative meets the design ranges of all other criteria successfully but not any of these ranges. However, the decision maker can assign a numerical value instead of an “infinitive” in order to make the selection of an alternative possible which meets all other criteria successfully, except the criterion having an “infinitive” value.

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MEASUREMENT OF LEVEL-OF-SATISFACTION OF DECISION MAKER IN INTELLIGENT FUZZY-MCDM THEORY: A GENERALIZED APPROACH

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Abstract: The earliest definitions of decision support systems (DSS) identify DSS as systems to support managerial decision makers in unstructured or semi-unstructured decision situations. They are also defined as a computer-based information systems used to support decision-making activities in situations where it is not possible or not desirable to have an automated system perform the entire decision process. This chapter aims to delineate measurement of level-of-satisfaction during decision making under an intelligent fuzzy environment. Before proceeding with the multi-criteria decision making model (MCDM), authors try to build a co-relation among DSS, decision theories, and fuzziness of information. The co-relation shows the necessity of incorporating decision makers' level-of-satisfaction in MCDM models. Later, the authors introduce an MCDM model incorporating different cost factor components and the said level-of-satisfaction parameter. In a later chapter, the authors elucidate an application as well as validation of the devised model. The strength of the proposed MCDM methodology lies in combining both cardinal and ordinal information to get eclectic results from a complex, multi-person and multi-period problem hierarchically.

Key words: Decision support system, level-of-satisfaction in MCDM

C. Kahraman (ed.), Fuzzy Multi-Criteria Decision Making.

1. INTRODUCTION

Nomenclature

- D Decision matrix
- A Pair-wise comparison matrix among criteria (m × n)
- m Number of criteria
- n Number of alternatives of the pair-wise comparison matrix
- η_{max} Principal eigen value of "A" matrix
- PV Priority vector
- I.I. Inconsistency index of "A" matrix
- R.I. Random inconsistency index of "A" matrix
- I.R. Inconsistency ratio of "A" matrix
- α Level of satisfaction of decision maker
- OFM Objective factor measure
- SFM Subjective factor measure
- OFC Objective factor cost
- SI Selection index
- γ Fuzzy parameter that measures the degree of vagueness; $\gamma = 0$ indicates crisp.

1.1 DSS and Their Components

Decision support systems (DSS) can be defined as computer-based information systems that aid a decision maker in making decisions for semi-structured problems. Numerous definitions to DSS exist. The earliest definitions of DSS (Gorry and Morton, 1977) identify DSS as systems to support managerial decision makers in unstructured or semi-unstructured decision situations. Ginzberg and Stohr (1981) propose DSS as "a computer-based information system used to support decision making activities in situations where it is not possible or not desirable to have an automated system performs the entire decision process." However, the most apt working definition is provided by Turban (1990). According to Turban (1990) "a DSS is an interactive, flexible, and adaptable computer based information system that utilizes decision rules, models, and model base coupled with a comprehensive database and the decision maker's own insights, leading to specific, implementable decisions in solving problems that would not be amenable to management science models per se. Thus, a DSS supports complex decision making and increases its effectiveness." Alter (2004) explores the assumption that stripping the word *system* from DSS, focusing on decision support, and using ideas related to the work system method might generate some interesting directions for research and practice. Some of these directions fit under the DSS umbrella, and some seem to be excluded because they are not directly related to a technical artifact called a DSS. Alter (2004) suggests that "decision support is the use of any plausible computerized or non-computerized means for improving sense making and/or decision making in a particular repetitive or non-repetitive business situation in a particular organization."

However, the main objectives of DSS can be stated as follows:

1. To provide assistance to decision makers in situations that are semi-structured,
2. To identify plans and potential actions to resolve problems,
3. To rank the solutions identified that can be implemented and provide a list of viable alternatives.

DSS attempts to bring together and focus several independent disciplines. These are as follows:

1. Operations research (OR),
2. Management science (MS),
3. Database technology,
4. Artificial intelligence (AI),
5. Systems engineering,
6. Decision analysis.

Artificial intelligence is a field of study that attempts to build software systems exhibiting near-human "intellectual" capabilities. Modern works on AI are focused on fuzzy logic, artificial neural networks (ANNs), and genetic algorithms (GAs). These works, when integrated with DSS, enhance the performance of making decisions. AI systems are used in creating intelligent models, analyzing models intelligently, interpreting results found from models intelligently, and choosing models appropriately for specific applications.

Decision analysis may be divided into two major areas. The first, descriptive analysis, is concerned with understanding how people actually make decisions. The second, normative analysis, attempts to prescribe how people should make decisions. Both are issues of concern to DSS. The central aim of decision analysis is improving decision making processes.

Decisions, in general, are classified into three major categories:

- Structured decisions,
- Unstructured decisions,
- Semi-structured decisions.

Structured decisions are those decisions where all steps of decision making are well structured. Computer code generation is comparatively easy for these types of decisions.

In unstructured decisions, none of the steps of decision making is structured. AI systems are being built up to solve the problems of unstructured decisions.

Semi-structured decisions comprise characteristics of structured and unstructured decisions.

The DSS framework contains two types of components, which may be used either individually or in tandem. The first component is a multi-objective programming (MOP) model, which employs mathematical programming to generate alternative mitigation plans. Typically, an MOP model must be formulated for the specific problem at hand, but once formulated, it can be solved on a computer using commercially available software. The second component is a multi-criteria decision making (MCDM) model, used for evaluating decision alternatives that have been generated either by the MOP model or by some other method. MCDM models are typically "shells" that can be applied to a wide range of problem types. A variety of MCDM methodologies exist, some of which are available in the form of commercial software. A manufacturing information system can also be used in conjunction with the DSS, for both managing data and for compiling decisions of alternative plans generated by the DSS.

1.2 Decision-Making Processes

Strategic, tactical, and operative decisions are made on the various aspects of business operations. The vision of an industrial enterprise must take into consideration the possible changes in its operational environment, strategies, and the leadership practices. Decision making is supported by analyses, models, and computer-aided tools. Technological advances have an impact on the business of industrial enterprises and on their uses of new innovations. Industrial innovations contribute to increased productivity and the diversification of production and products; they help to create better, more challenging jobs and to minimize risks.

Long-term decisions have an impact on process changes, functional procedures and maintenance and also on safety, performance, costs, human factors and organisations. Short-term decisions deal with daily actions and their risks. Decision-making is facilitated by an analysis that incorporates a classification of one's own views, calculating numerical values, translating the results of analysis into concrete properties and a numerical evaluation of the properties. One method applied for this purpose is the Analytic Hierarchy Process (AHP) model (Saaty, 1990). This model, which has many features in common with the other MCDMs applied in the current research work, is suited for manufacturing decision making processes that aim at making the correct choices in both the short and the long term.

1.3 MCDM

According to Agrell (1995) MCDM offers the methodology for decision making analysis when dealing with multiple objectives. This may be the case when the success of the application depends on the properties of the system, the decision maker, and the problem. Problems with engineering design involve multiple criteria: the transformation of resources into artifacts, a desire to maximize performance, and the need to comply with specifications.

The MCDM methodology can be used to increase performance and to decrease manufacturing costs and delays of enterprises. The Multiple-Criteria Decision Support System (MC-DSS) uses the MCDM methodology and ensures mathematical efficiency. The system employs graphical presentations and can be integrated with other design tools. Modeling and analyzing complex systems always involve an array of computational and conceptual difficulties, whereas a traditional modeling approach is based primarily on simulation and concepts taken from control theory.

The strength of the MCDM lies in the systematic and quantitative framework it offers to support decision making. Comprehensive tuning or parametric design of a complex system requires elaboration on using the modeling facilities of system dynamics and on the interactive decision making support of the MCDM.

Most experienced decision makers do not rely on a theory to make their decisions because of cumbersome techniques involved in the process of making decisions. But analytic decision making is of tremendous value when the said analytic process involves simple procedures and is accessible to the lay user as well as it possesses meaningful scientific justification of the highest order (Saaty, 1994).

The benefits of descriptive analytical approaches for decision making are as follows (Saaty, 1994):

1. To permit decision makers to use information relating to decision making in a morphological way of thoroughly modeling the decision and to make explicit decision makers' tactical knowledge;
2. To permit decision makers to use judgments and observations in order to surmise relations and strengths of relations in the flow of interacting forces moving from the general to the particular and to make predictions of most likely outcomes;
3. To enable decision makers to incorporate and trade off attribute values;
4. To enable decision makers to include judgments that result from intuition, day-to-day experiences, as well as those that result from logic;
5. To allow decision makers to make gradual and more thorough revisions and to combine the conclusions of different people studying the same problem in different places.

1.4 Information *vis-à-vis* MCDM Theories

Information is a system of knowledge that has been transformed from raw "data" into some meaningful form. Data are the raw materials for information. Data are also expressions of "events." Information has value in current or prospective decision making at a specified time and place for taking appropriate "action" resulting in evaluation of "performance." In this context attention is drawn to Figure 1. The terms "data" and "information" are often used interchangeably, but there is a distinction in that. Data are processed to provide information, and the information is related to decision making (Davis, 1974). A schematic diagram illustrating relationship between data and information is shown in Figure 2. If there is no need for making decisions, information would be unnecessary.

Information is the currency of the new economy. Yet most real-world cases lack the means to effectively organize and distribute the information their employees need to make quick, smart business decisions. A structured, personalized, self-serve way to access information and collaborate across departmental and geographical boundaries provides the basic needs for making a good decision.

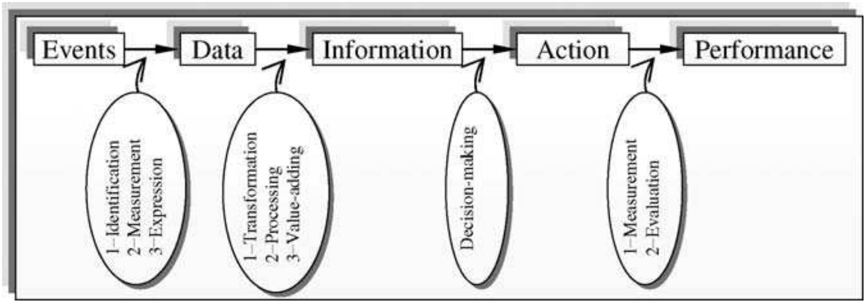

Figure 1. Generation and utilization of information

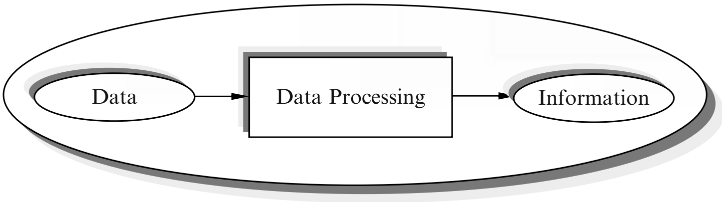

Figure 2. Converting raw data by an information system into useful information

1.5 Hidden Parameters in Information

1.5.1 Uncertainty in Information

Uncertainty permeates understanding of the real world. The purpose of information systems is to model the real world. Hence, information systems must be able to deal with uncertainty.

Many information systems include capabilities for dealing with some kinds of uncertainty. For example, database systems can represent missing values, information retrieval systems can match information to requests using a "weak" matching algorithm, and expert systems can represent rules that are known to be true only for "most" or "some" of the time. By and large, commercial information systems (e.g., database systems, information retrieval systems, or expert systems) have been slow to incorporate capabilities for dealing with uncertainty.

Uncertainty also has a long history of being associated with decision making research as Harris (1998) notes:

> Decision making is the process of sufficiently reducing uncertainty and doubt about alternatives to allow a reasonable choice to be

> made from among them. This definition stresses the information gathering function of decision making. It should be noted here that uncertainty is reduced rather than eliminated. Very few decisions are made with absolute certainty because complete knowledge about all the alternatives is seldom possible.

Researchers in various fields have also been concerned with the relationship between uncertainty and information seeking. In information science, the idea of uncertainty underlies all aspects of information seeking and searching. Kuhlthau (1993) has proposed uncertainty as a basic principle for information seeking, defining uncertainty as "a cognitive state which commonly causes affective symptoms of anxiety and lack of confidence." And, drawing on her research, she notes that, "Uncertainty and anxiety can be expected in the early stages of the information search process.… Uncertainty due to a lack of understanding, a gap in meaning, or a limited construct initiates the process of information seeking."

One of the biggest challenges for a manufacturing decision maker is the degree of uncertainty in the information that he or she has to process. In making some decisions, this is especially obvious when experts in the same area provide conflicting opinions on the attributes meant for making decisions. Disagreement among experts making decisions results in conflicting effects information. The decision maker is likely to place increased importance on the source of the information. This in itself is not surprising, but the battle of the credentials that follows perhaps is. There seems to be a danger that the may come to rely on the reputation of an expert, rather than on ensuring thorough scrutiny of the information that he or she has provided.

Actors in the decision making process may use uncertainty in the *effects*, and information as a means to promote their attributes. A proponent can try to downplay the *effects* of a development because they may not occur, whereas those in opposition may attempt to stall a project claiming that the disputed *effects* are likely to happen and are serious in nature. The decision maker is then left with the difficult task of navigating these disparities to come to a decision. In particular in the face of uncertainty, there seems to be a human tendency to make personal observations the deciding factor.

1.5.1.1 Sources of Uncertainty

Uncertainties are solely due to the unavailability of "perfect" information. Uncertainty might result from using unreliable information sources, for example, faulty reading instruments, or input forms that have been filled out incorrectly (intentionally or inadvertently). In other cases, uncertainty is a result of system errors, including input errors, transmission "noise," delays in processing update transactions, imperfections of the system software, and corrupted data owing to failure or sabotage. At times, uncertainty is the unavoidable result of information gathering methods that require estimation or judgment.

In other cases, uncertainty is the result of restrictions imposed by the model. For example, if the database schema permits storing at most two occupations per employee, descriptions of occupation would exhibit uncertainty. Similarly, the sheer volume of information that is necessary to describe a real-world object might force the modeler to turn to approximation and sampling techniques.

1.5.1.2 Degree of Uncertainty

The relevant information that is available in the absence of certain information may take different forms, each exhibiting a different level of uncertainty. Uncertainty is highest when the mere existence of some real-world object is in doubt. The simplest solution is to ignore such objects altogether. This solution, however, is unacceptable if the model claims to be closed world (i.e., objects not modeled do not exist).

Uncertainty is reduced somewhat when each element is assigned a value in a prescribed range, to indicate the certainty that the modeled object exists. When the element is a fact, this value can be interpreted as the confidence that the fact holds; when it is a rule, this value can be interpreted as the strength of the rule (percent of cases where the rule applies).

Now it is assumed that "existence" is assured, but some or all of the information with which the model describes an object is unknown. Such information has also been referred to as incomplete, missing, or unavailable.

Uncertainty is reduced when the information that describes an object is known to come from a limited set of alternatives (possibly a range of values). This uncertainty is referred to as disjunctive information. Note that when the set of alternatives is simply the entire "universe," this case reverts to the previous (less informative) case.

Uncertainty is reduced even more when each alternative is accompanied by a number describing the probability that it is indeed the true description (and the sum of these numbers for the entire set is 1). In this case, the uncertain information is probabilistic. Again, when the probabilities are unavailable, probabilistic information becomes disjunctive information.

Occasionally, the information available to describe an object is descriptive rather than quantitative. Such information is often referred to as fuzzy or vague information.

1.5.1.3 Vagueness in Information

Russell (1923) attributes vagueness to being mostly a problem of language. Of course, language is part of the problem, but it is not the main problem. There would still be vagueness even if we had a very precise, logically structured, language. The principal source of vagueness seems to be in making discreet statements about continuous phenomenon. According to Russell (1923), “Vagueness in a cognitive occurrence is a characteristic of its relation to that which is known, not a characteristic of the occurrence in itself.” Russell (1923) adds, “Vagueness, though it applies primarily to what is cognitive, is a conception applicable to every kind of representation.”

Surprisingly, Wells (1908) was among the first to suggest the concept of vagueness:

> Every species is vague, every term goes cloudy at its edges, and so in my way of thinking, relentless logic is only another name for stupidity for a sort of intellectual pigheadedness. If you push a philosophical or metaphysical enquiry through a series of valid syllogisms never committing any generally recognized fallacy you nevertheless leave behind you at each step a certain rubbing and marginal loss of objective truth and you get deflections that are difficult to trace, at each phase in the process. Every species waggles about in its definition, every tool is a little loose in its handle, every scale has its individual.

In real-world problems there is always a chance of getting introduced to the vagueness factor when information deals in combination with both cardinal and ordinal measures. It should always be remembered that reduction of vagueness is to be addressed in a situation where decision alternatives are well inter-related and have both cardinal and ordinal criteria for selection.

1.5.1.4 Sources of Vagueness

Linguistic expressions in classic decision making processes incorporate unquantifiable, imperfect, nonobtainable information and partially ignorant facts. Data combining both ordinal and cardinal preferences in real-world decision making problems are highly unreliable and both contain a certain degree of vagueness. Crisp data often contains some amount of vagueness and, therefore, need the attention of decision makers in order to achieve a lesser degree of vagueness inherent.

The purpose of decision making processes is best served when imprecision is communicated as precisely as possible but no more precisely than warranted.

2. PRIOR WORKS ON FUZZY-MCDM FOR SELECTING BEST CANDIDATE-ALTERNATIVE

The available literature on MCDM tackling fuzziness is as broad as it is diverse. Literature contains several proposals on how to incorporate the inherent uncertainty as well as the vagueness associated with the decision maker's knowledge into the model (Arbel, 1989; Arbel and Vargas, 1990; Banuelas and Antony, 2004; Saaty and Vargas, 1987). The analytic hierarchy process (AHP) (Saaty, 1980 and 1990) literature, in this regard, is also vast.

There has been a great deal of interest in the application of fuzzy sets to the representation of fuzziness and uncertainty in management decision models (Buckley, 1988; Chen and Hwang, 1982; Ghotb and Warren, 1995; Gogus and Boucher, 1997; Van Laarhoven and Pedrycz, 1983; Liang and Wang, 1994; Lai and Hwang, 1994; Zimmerman, 1976, 1987). Some approaches were made to handle the uncertainties of MCDM problems. Bellman and Zadeh (1970) have shown fuzzy set theory's applicability to the MCDM study. Yager and Basson (1975) and Bass and Kwakernaak (1977) have introduced maximin and simple additive weighing model using the membership function (MF) of the fuzzy set. Most of the recent literature is filled with mathematical proofs.

A decision maker needs an MCDM assessment technique in regard to its fuzziness that can be easily used in practice. An approach was taken earlier by Marcelloni and Aksit (2001). Their aim was to model inconsistencies through the application of fuzzy logic-based techniques. Boucher and Gogus (2002) examined certain characteristics of judgment elicitation instruments appropriate to fuzzy MCDM. In their work the fuzziness was measured using a gamma function.

By defining a decision maker's preference structure in fuzzy linear constraint (FLC) with soft inequality, one can operate the concerned fuzzy optimization model with a modified *S*-curve smooth MF to achieve the desired solution (Watada, 1997). One form of logistic MF to overcome difficulties in using a linear membership function in solving a fuzzy decision making problem was proposed by Watada (1997). However, it is expected that a new form of logistic membership function based on nonlinear properties can be derived, and its flexibility in fitting real-life problem parameters can be investigated. Such a formulation of a nonlinear logistic MF was presented in this work, and its flexibility in taking up the fuzziness of the parameter in a real-life problem was demonstrated.

Carlsson and Korhonen (1986) have illustrated, through an example, the usefulness of a formulated MF, viz., an exponential logistic function. Their illustrated example was adopted to test and compare a nonlinear MF (Lootsma, 1997). Such an attempt using the said validated nonlinear MF and comparing the results was made by Vasant et al. (2005). Comprehensive tests based on a real-life industrial problem have to be undertaken on the newly developed membership function in order to prove further its applicability in fuzzy decision making (Vasant, 2003; Vasant et al., 2002; 2005). To test the newly formulated MF in problems as stated above, a software platform is essential. In this work MATLAB has been chosen as the software platform using its M-file for greater flexibility.

In the past, studies on decision making problems were considered on the bipartite relationship of the decision maker and analyst (Tabucanon, 1996). This is with the assumption that the implementers are a group of robots that are programmed to follow instructions from the decision maker. This notion is now outdated. Now a tripartite relationship is to be considered, as shown on Figure 3, where the decision maker, the analyst, and the implementer will interact in finding a fuzzy satisfactory solution in any given fuzzy system. This is because the implementers are human beings, and they have to accept the solutions given by the decision maker to be implemented under a turbulent environment.

In case of tripartite fuzzy systems, the decision maker will communicate and describe the fuzzy problem with an analyst. Based on

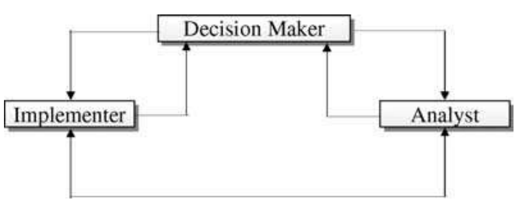

Figure 3. Tripartite relationship for MCDM problems

the data that are provided by the decision maker, the analyst will formulate MFs, solve the fuzzy problems, and provide the solution back to the decision-maker. After that, the decision maker will provide the fuzzy solution with a trade off to the implementer for implementation. An implementer has to interact with decision maker to obtain an efficient and highly productive fuzzy solution with a certain degree of satisfaction. This fuzzy system will eventually be called a high productive fuzzy system (Rommelfanger, 1996). A tripartite relationship, decision maker−analyst−implementer, is essential to solve any industrial problem.

The following criticisms of the existing literatures, in general, are made after a study of the existing vast literature on the use of various types of MFs in finding out fuzziness patterns of MCDM methodologies:

1. Data combining both ordinal and cardinal preferences contain non-obtainable information and partially ignorant facts. Both ordinal and cardinal preferences contain a certain degree of fuzziness and are highly unreliable, unquantifiable and imperfect.
2. Simplified fuzzy MFs, viz., trapezoidal and triangular and even gamma functions, are not able to bring out real-world fuzziness patterns in order to elucidate a degree of fuzziness inherent in the MCDM model.
3. Level-of-satisfaction of the decision makers should be judged through a simple procedure while making decisions through MCDM models.
4. An intelligent tripartite relationship among the decision maker, analyst and implementer is essential, in conjunction to a more flexible MF design, to solve any real-world MCDM problem.

Among many diversified objectives of the current work, one objective is to find out fuzziness patterns of the candidate-alternatives having disparate level-of-satisfaction in MCDM model. Relationships among the degree of fuzziness, level-of-satisfaction and the selection-indices of the MCDM model guide decision makers under a tripartite fuzzy environment in obtaining their choice tradeoff with a predetermined allowable imprecision.

Another objective of the current work is to provide a robust, quantified monitor of the level-of-satisfaction among decision makers and to calibrate these levels of satisfaction against decision makers' expectations. Yet another objective is to provide a practical tool for further assessing the impact of different options and available courses of action.

3. COMPONENTS OF THE MCDM MODEL

The proposed MCDM model considers a fuzziness pattern in disparate level-of-satisfaction of the decision maker. The model outlines a MF for evaluating degree of fuzziness hidden in the Eq. (1). AHP provides the decision maker's with a vector of priorities (PV) to estimate the expected utilities of each candidate-FMS.

A mathematical model was proposed by Bhattacharya et al. (2004, 2005) to combine cost factor components with the importance weightings found from AHP. The governing Eq. of the said model is:

OFM = Objective factor measure,
OFC = Objective factor cost,
SFM = Subjective factor measure,
SI = Selection index,
α = Objective factor decision weight,
n = Finite number of candidate-alternative.

$$\mathrm{SI}_i = \left[\left(\alpha \times \mathrm{SFM}_i\right) + \left(1-\alpha\right) \times \mathrm{OFM}_i\right] \tag{1}$$

where

$$\mathrm{OFM}_l = \frac{1}{\mathrm{OFC}_l \times \sum_{l=1}^{n} \mathrm{OFC}^l} \tag{2}$$

In the said model, AHP plays a crucial role. AHP is an MCDM method, and it refers to making decisions in the presence of multiple, usually conflicting, criteria. A criterion is a measure of effectiveness. It is the basis for evaluation. Criteria emerge as a form of attributes or objectives in the actual problem setting. In reality, multiple criteria usually conflict with each other. Each objective/attribute has a different unit of measurement. Solutions to the problems by AHP are either to design the best alternative or to select the best one among the previously specified finite alternatives.

For assigning the weights to each of the attributes as well as to the alternative processes for constructing the decision matrix and pair-wise comparison matrices, the phrase like "much more important" is used to extract the decision maker's preferences. Saaty (1990) gives an intensity scale of importance (refer to Table 1) and has broken down the importance ranks.

Table 1. The Nine-Point Scale of Pair-Wise Comparison

Intensity scale	Interpretation
1	Equally important
3	Moderately preferred
5	Essentially preferred
7	Very strongly preferred
9	Extremely preferred
2, 4, 6, 8	Intermediate importance between two adjacent judgments

In AHP the decision matrix is always a square matrix. Using the advantage of properties of eigenvalues and eigenvectors of a square matrix, the level of inconsistency of the judgmental values assigned to each elements of the matrix is checked.

In this chapter the proposed methodology is applied to calculate the priority weights for functional, design factors and other important attributes by eigenvector method for each pair-wise comparison matrix. Next, global priorities of various attributes rating are found by using AHP. These global priority values are used as SFM in Eq. (1). The pair-wise comparison matrices for five different factors are constructed on the basis of Saaty's nine-point scale (refer to Table 1). The objective factors, i.e., OFM, and OFC are calculated separately by using cost factor components.

In the mathematical modeling for finding the SFM_i values, decomposition of the total problem (factor-wise) into smaller sub-problems has been done. This is done so that each sub-problem can be analyzed and appropriately handled with practical perspectives in terms of data and information. The objective of decomposition of the total problem for finding out the SFM values is to enable a pair-wise comparison of all the elements on a given level with respect to the related elements in the level just above.

The proposed algorithm consists of a few steps of calculations. Prior to the calculation part, listing of the set of candidate-alternatives is carried out. Next, the cost components of the candidate-alternatives are quantified. Factors, on which the decision making is based, are identified as intrinsic and extrinsic. A graphical representation depicting the hierarchy of the problem in terms of overall objective, factors, and number of alternatives is to be developed. Next follows the assigning of the judgmental values to the factors as well as to the candidate-alternatives to construct the decision matrix and pair-wise comparison matrices, respectively.

A decision matrix is constructed by assigning weights to each factor based on the relative importance of its contribution according to a nine-point scale (refer to Table 1). Assigning the weights to each candidate-alternative for each factor follows the same logic as that of the decision matrix. This matrix is known as a pair-wise comparison matrix. The PV values are determined then for both the decision and the pair-wise comparison matrices. The $\eta_{\max}$ for each matrix may be found by multiplication of the sum of each column with the corresponding PV value and subsequent summation of these products.

There is a "check" in the judgmental values given to the decision and pair-wise comparison matrices for revising and improving the judgments. If I.R. is greater than 10%, the values assigned to each element of the decision and pair-wise comparison matrices are said to be inconsistent. For I.R. < 10%, the level of inconsistency is acceptable. Otherwise the level of inconsistency in the matrices is high and the decision maker is advised to revise the judgmental values of the matrices to produce more consistent matrices. It is expected that all the comparison matrices should be consistent. But the very root of the judgment in constructing these matrices is the human being. So, some degree of inconsistency of the judgments of these matrices is fixed at 10%. Calculation of I.R. involves I.I., R.I., and I.R. These matrices are evaluated from Eqs. (3), (4) and (5) respectively.

$$\text{I.I.} = \frac{(\eta_{\max} - n)}{(n - 1)} \tag{3}$$

$$\text{R.I.} = \frac{[1.98 \times (n - 2)]}{n} \tag{4}$$

$$\text{I.R.} = \frac{\text{I.I.}}{\text{R.I.}} \tag{5}$$

The OFM_i values are determined by Eq (6).

$$\text{OFM}_i = [\text{OFC}_i \times \sum_{i=1}^{n} \frac{1}{\text{OFC}_i}]^{-1} \tag{6}$$

The SFM_i values are the global priorities for each candidate-alternative. SFM_i may be found by multiplying each of the decision matrix PV values to each of the PV value of each candidate-alternative for each factor. Each product is then summed up for each alternative to get SFM_i.

For an easy demonstration of the proposed fuzzified MCDM model, efforts for additional fuzzification are confined assuming that differences in judgmental values are only 5%. Therefore, the upper bound and lower bound of SFM_i as well as SI_i indices are to be computed within a range of 5% of the original values. In order to avoid complexity in delineating the technique proposed hereinbefore, we have considered the 5% measurement. One can fuzzify the SFM_i values from the very beginning of the model by introducing a modified *S*-curve MF in AHP, and the corresponding fuzzification of SI_i indices can also be carried out using the holistic approach used in Eq. (1). The set of candidate-alternatives are then ranked according to the descending order of SI_i indices (refer to Eq. 7).

$$\tilde{LSI}_i\Big|_{\alpha=\alpha_{SFM_i}} = LSI_L + \left(\frac{LSI_U - LSI_L}{\gamma}\right)\ln\frac{1}{C}\left(\frac{A}{\alpha_{LSI_i}}-1\right) \tag{7}$$

In this work, a monotonically nonincreasing logistic function has been used as a membership function:

$$f(x) = \frac{B}{1+Ce^{\gamma x}} \tag{8}$$

where α is the level-of-satisfaction of the decision maker; *B* and *C* are scalar constants; and γ, $0<\gamma<\infty$ is a fuzzy parameter that measures the degree of vagueness (fuzziness), wherein $\gamma = 0$ indicates crisp. Fuzziness becomes highest when $\gamma \to \infty$.

The generalized logistic membership function is defined as

$$f(x) = \begin{cases} 1 & x < x_L \\ \dfrac{B}{1+Ce^{\gamma x}} & x_L < x < x_U \\ 0 & x > x_U \end{cases} \tag{9}$$

To fit into the MCDM model in order to sense its degree of fuzziness, the Eq. (9) is modified and redefined as follows:

$$\mu(x) = \begin{cases} 1 & x < x^a \\ 0.999 & x = x^a \\ \dfrac{B}{1+Ce^{\gamma x}} & x^a < x < x^b \\ 0.001 & x = x^b \\ 0 & x > x^b \end{cases} \tag{10}$$

In Eq. (10) the membership function is redefined as $0.001 \le \mu(x) \le 0.999$. This range is selected because in real-world situations the workforce need not be always 100% of the requirement. At the same time the workforce will not be 0%. Therefore, there is a range between x_0 and x_1 with $0.001 \leq \mu(x) \leq 0.999$. This concept of range of $\mu(x)$ is used in this chapter.

Choice of the level-of-satisfaction of the decision maker, i.e., α, is an important issue. It is the outcome of the aggregate decision by the design engineer, production engineer, maintenance engineer, and capital investor of a manufacturing organization. However, the selection of a candidate-alternative may give different sets of results for different values of α for the same attributes and cost factor components. That's why the proposed model includes fuzzy-sensitivity plots to analyse the effect of α as well as the degree of fuzziness, γ, in the candidate-alternative selection problem.

4. FORMULATION OF THE INTELLIGENT FUZZIFIED MCDM MODEL

4.1 Membership Function

There are 11 in-built membership functions in the MATLAB fuzzy toolbox. In the current study, a modified version of No. 7 MF has been used. All the built-in MF includes 0 and 1. In the current work, 0 and 1 have been excluded and the *S*-shaped membership function has been extensively modified accordingly.

As mentioned by Watada (1997), a trapezoidal MF will have some difficulties such as degeneration, i.e., some sort of deterioration of solution, while introducing fuzzy problems. In order to solve the issue of degeneration, we should employ a non linear logistic function such as a tangent hyperbolic that has asymptotes at 1 and 0.

In the current work, we employ the logistic function for the nonlinear membership function as given by

$$f(x) = \frac{B}{1 + Ce^{\gamma x}} \tag{11}$$

where B and C are scalar constants and γ, $0 < \gamma < \infty$ is a fuzzy parameter that measures the degree of vagueness, wherein $\gamma = 0$ indicates crisp. Fuzziness becomes highest when $\gamma \to \infty$.

Eq. (11) will be of the form as indicated by Figure 4 when $0 < \gamma < \infty$.

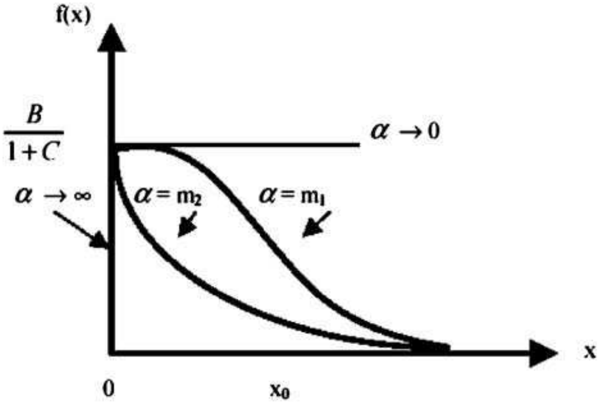

Figure 4. Variation of logistic MF with respect to fuzzy parameter, γ (where $m_2 > m_1$)

The reason why we use this function is that the logistic MF has a similar shape as that of the tangent hyperbolic function employed by Leberling (1981) but it is more flexible (Bells, 1999) than the tangent hyperbola. It is also known that a trapezoidal MF is an approximation to a logistic function. Therefore, the logistic function is very much considered an appropriate function to represent a vague goal level. This function is found to be very useful in making decisions and in implementation by the decision maker and implementer (Lootsma, 1997; Zimmerman, 1985; 1987).

Moreover, to avoid linearity in the real-life application problems, especially in industrial engineering problems, a non linear function such as modified MF can be used. This MF is used when the problems and its solutions are independent (Varela and Riberio, 2003). It should be emphasized that some nonlinear MFs such as *S*-curve MFs are much more desirable for real-life application problems than that of linear MFs.

The logistic function, Eq. (11), is a monotonically nonincreasing function, which will be employed as a fuzzy MF. This is very important because, due to an uncertain environment the availability of the variables are represented by the degree of fuzziness.

The said MF can be shown to be non increasing as

$$\frac{df}{dx} = \frac{BC\alpha e^{\gamma x}}{(1+Ce^{\gamma x})^2} \tag{12}$$

An MF is flexible when it has vertical tangency, an inflexion point, and asymptotes. Since *B*, *C*, γ, and x are all greater than zero, $\frac{df}{dx} \le 0$. Furthermore it can be shown that Eq. (11) has asymptotes at *f(x)* = 0 and *f(x)* = 1 at appropriate values of *B* and *C*. This implies:

$$-\lim_{x\to\infty}\frac{df}{dx} = 0 \text{ and } \lim_{x\to 0}\frac{df}{dx} = 0$$

These asymptotes can be proved as follows.

From Eq. (12), one gets

$$\lim_{x\to\infty}\frac{df}{dx} = -\frac{\infty}{\infty}.$$

Therefore, using L'hopital's rule, one obtains

$$\lim_{x\to\infty}\frac{df}{dx} = -\lim_{x\to\infty}\frac{B\gamma}{2(1+Ce^{\gamma x})} = 0 \tag{13}$$

As $x \to 0$, the situation is less vague and hence $\gamma \to 0$.
From Eq. (12), one gets

$$\lim_{x\to 0}\frac{df}{dx} = -\frac{BC\gamma}{(1+C)^2} = 0, \text{ when } \gamma \to 0 \tag{14}$$

In addition to the above equation, it can be shown that the logistic function Eq. (11) has a vertical tangent at $x = x_0$, x_0 is the point where $f(x_0) = 0.5$.

Furthermore it can also be shown that the said logistic function has a point of inflexion at $x = x_0$, such that $f''(x_0) = \infty$, with $f''(x)$ being the second derivative of *f(x)* with respect to *x*. An MF of *S*-curve nature, in contrast to linear function, exhibits the real-life problem.

The generalized logistic MF is defined as

$$f(x)=\begin{cases}1 & x<x_L\\ \dfrac{B}{1+Ce^{\gamma x}} & x_L<x<x_U\\ 0 & x>x_U\end{cases} \tag{15}$$

The *S*-curve MF is a particular case of the logistic function defined in Eq. (15). The said *S*-curve MF has specific values of *B*, *C* and γ. The logistic function as defined in Eq. (11) was indicated as an *S*-curve MF by Zadeh (1971; 1975).

4.2 Design of Modified, Flexible *S*-curve MF

To fit into the MCDM model in order to sense its degree of fuzziness, Eq. (15) is modified and redefined as follows and illustrated in Figure 5.

$$\mu(x)=\begin{cases}1 & x<x^a\\ 0.999 & x=x^a\\ \dfrac{B}{1+Ce^{\gamma x}} & x^a<x<x^b\\ 0.001 & x=x^b\\ 0 & x>x^b\end{cases} \tag{16}$$

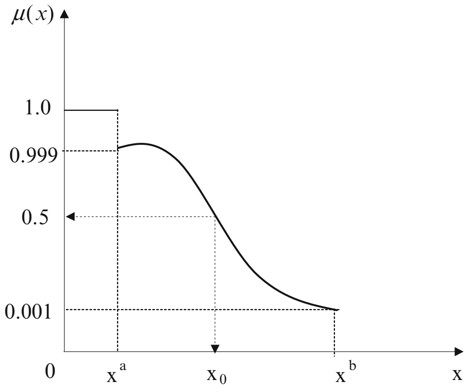

Figure 5. Modified *S*-curve membership function

We rescale the x-axis as $x^a = 0$ and $x^b = 1$ in order to find the values of B, C, and γ. Nowakowska (1977) has performed such a rescaling in his work on the social sciences.

The values of B, C, and γ are obtained from Eq. (16) as

$$B = 0.999\ (1 + C) \tag{17}$$

$$\frac{B}{1 + Ce^{\gamma}} = 0.001. \tag{18}$$

By substituting Eq. (17) into Eq. (18), one gets

$$\frac{0.999\left(1 + C\right)}{1 + Ce^{\gamma}} = 0.001. \tag{19}$$

Rearranging Eq. (19), one gets

$$\gamma = \ln \frac{1}{0.001}\left(\frac{0.998}{C} + 0.999\right). \tag{20}$$

Since B and γ depend on C, one requires one more condition to get the values for B, C, and γ.

Let, when $x_0 = \dfrac{x^a + x^b}{2}$, $\mu(x_0) = 0.5$.

Therefore,

$$\frac{B}{1 + Ce^{\frac{\gamma}{2}}} = 0.5, \tag{21}$$

and hence

$$\gamma = 2 \ln\left(\frac{2B - 1}{C}\right). \tag{22}$$

Substituting Eq. (20) and Eq. (21) into Eq. (22), we obtain

$$2\ \ln\ \left(\frac{2(0.999)(1+C)-1}{C}\right) = \ln\ \frac{1}{0.001}\ \left(\frac{0.998}{C}\ +\ 0.999\right) \tag{23}$$

$$\left(0.998+1.998C\right)^2 = C\left(998+999C\right) \text{ which in turn yields} \tag{24}$$

Eq. (24) is solved and it is found that

$$C = \frac{-994.011992\ \pm\ \sqrt{988059.8402\ +\ 3964.127776}}{1990.015992} \tag{25}$$

Since C has to be positive, Eq. (22) gives $C = 0.001001001$, and from Eqs. (17) and (22), one gets $B = 1$ and $\gamma = 13.81350956$.

Thus, it is evident from the preceding sections that the flexible, modified *S*-curve MF can be more easily handled than other nonlinear MFs such as the tangent hyperbola. The linear MF such as the trapezoidal MF is an approximation from a logistic MF and is based on many idealistic assumptions. These assumptions contradict the realistic real-world problems.

Therefore, the S-curve MF is considered to have more suitability in sensing the degree of fuzziness in the fuzzy-uncertain judgmental values of a decision maker. The modified *S*-curve MF changes its shape according to the fuzzy judgmental values of a decision maker and therefore, a decision maker finds it suitable to apply his/her strategy to MCDM problems using these judgmental values.

Thus the proposed *S*-shaped membership function is flexible due to its following characteristics:

(i) $\mu(x)$ is continuous and strictly monotonously nonincreasing;

(ii) $\mu(x)$ has lower and upper asymptotes at $\mu(x) = 0$ and $\mu(x) = 1$ as $x \to \infty$ and $x \to 0$, respectively;

(iii) $\mu(x)$ has inflection point at

$$x_0 = \frac{1}{\alpha}\ln\left(2+\frac{1}{C}\right) \text{ with } A = 1+C$$

The fuzzy intelligence of the proposed MCDM model is incorporated under a tripartite environment. A fuzzy rule-based decision (*if−then* rule) is incorporated in the algorithm to sense the fuzziness patterns under a disparate level-of-satisfaction of the decision maker. The aim is to produce a rule that works well on previously unseen data.

In the next chapter it will be demonstrated how to compute the degree of fuzziness and level-of-satisfaction, and a correlation among degree of fuzziness having a disparate level-of-satisfaction and the selection indices will also be elucidated to guide the decision maker in selecting the best candidate-alternative under an unstructured environment.

5. CONCLUSION

The proposed MCDM model shows how to measure a parameter called “level-of-satisfaction” of the decision maker while making any kind of decision. “Level-of-satisfaction” is a much-quoted terminology in classic as well as modern economics. To date, we are not aware of any reported research work in which level-of-satisfaction has been measured with a rigorous mathematical logic. The proposed model is a one-of-a-kind solution to incorporate the “level-of-satisfaction” of decision maker. Another solution can also be made with many sophisticated tools, like some approximation tool using neuro-fuzzy hybrid models.

The strength of the proposed MCDM methodology lies in combining both cardinal and ordinal information to get eclectic results from a complex, multi-person, and multi-period problem hierarchically. The methodology proposed in this chapter is very useful in quantifying the intangible factors in a good manner and in finding out the best among the alternatives depending on their cost factors. Contrary to the traditional way of selection using discounted cash flow (DCF), this methodology is a sound alternative to apply under an unstructured environment.

There may be some weaknesses due to the nonavailability of experts’ comments, i.e., judgmental values. Comparison among various similar types of systems is the opportunity of the proposed model. An underlying threat is associated with the proposed model that a illogical decisions and mis-presentation of experts comments may lead to a wrong decision.

The MCDM methodology proposed in this chapter assumes that the decision is made under a fuzzy environment. A comparative study by accommodating different measures of uncertainty and risk in the MADM methodology may also be made to judge the best-suited measure of uncertainty. A knowledge-based system may be developed based on the modified AHP.

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FMS SELECTION UNDER DISPARATE LEVEL-OF-SATISFACTION OF DECISION MAKING USING AN INTELLIGENT FUZZY-MCDM MODEL

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Abstract: This chapter outlines an intelligent fuzzy multi-criteria decision-making (MCDM) model for appropriate selection of a flexible manufacturing system (FMS) in a conflicting criteria environment. A holistic methodology has been developed for finding out the “optimal FMS” from a set of candidate-FMSs. This method of trade-offs among various parameters, viz., design parameters, economic considerations, etc., affecting the FMS selection process in an MCDM environment. The proposed method calculates the global priority values (GP) for functional, design factors and other important attributes by an eigenvector method of a pair-wise comparison. These GPs are used as subjective factor measures (SFMs) in determining the selection index (SI). The proposed fuzzified methodology is equipped with the capability of determining changes in the FMS selection process that results from making changes in the parameters of the model. The model achieves balancing among criteria. Relationships among the degree of fuzziness, level-of-satisfaction and the SIs of the MCDM methodology guide decision makers under a tripartite fuzzy environment in selecting their choice of trading-off with a predetermined allowable fuzziness. The measurement of level-of-satisfaction during making the appropriate selection of FMS is carried out.

Key words: FMS, intelligent fuzzy MCDM, global priority, sensitivity analysis, selection indices

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1. INTRODUCTION

A flexible manufacturing system (FMS) is a set of integrated computer-controlled, automated material handling equipments and numerical-controlled machine tools capable of processing a variety of part types. Due to the competitive advantages like flexibility, speed of response, quality, reduction of lead-time, reduction of labour etc., FMSs are now gaining popularity in industries.

Today's manufacturing strategy is purely a choice of alternatives. The better the choice, more will be the productivity as well as the profit maintaining quality of product and responsiveness to customers. In this era of rapid globalization, the overall objective is to purchase a minimum amount of capacity (i.e., capital investment) and utilize it in the most effective way. Although FMS is an outgrowth of existing manufacturing technologies, its selection is not often studied. It has been a focal point in manufacturing–related research since the early 1970s. FMS provides a low inventory environment with unbalanced operations unique to the conventional production environment. The process design of an FMS consists of a set of crucial decisions that are to be made carefully. It requires decision making, e.g., selection of a CNC machine tool, material handling system, or product mix. The selection of a FMS thus requires trading-off among the various parameters of the FMS alternatives. The selection parameters are conflicting in nature. High-quality management is not enough for dealing with the complex and ill-structured factors that are conflicting in nature (Buffa, 1993). Therefore, there is a need for sophisticated and applicable technique to help the decision makers for selecting the proper FMS in a manufacturing organization.

The authors, thus, propose a DSS methodology, for appropriate FMS selection, that trade off among some intangible criteria as well as cost factors to get the maximum benefit out of these conflicting-in-nature criteria. There have been many contributors to the literature on selection of proper FMS. A selective review of some of the relevant works in this area is give here. Kaighobadi and Venkatesh (1994) presented an overview and survey of research in FMSs. They also presented a definition of FMSs. Chen et al. (1998) investigated the relationship between flexibility measurements and system performance in the flexible manufacturing systems environment. The authors suggested several alternative measures for the assessment of machine flexibility and routing flexibility—two of the most important flexibility types discussed in the literature. Nagarur (1992) showed that computer integration and flexibility of the system were the two critical factors of FMS. Eight different types of flexibility were proposed by Browne et al. (1984). Each of these flexibilities contributes to overall flexibility of the system to cope with possible changes in demand structure. In addition to machine, process, product, routing, volume, expansion, operation and production flexibility as described by Browne et al. (1984). Barad and Sipper (1988) introduced another classification, i.e., transfer flexibility. Buzacott and Mandelbaum (1985) defined flexibility as the ability of a manufacturing system to cope with changing circumstances. High-level flexibility enables a manufacturing firm to provide faster response to market changes maintaining high product quality standards (Gupta and Goyal, 1989).

Flexible manufacturing provides an environment where integration effects cannot be eliminated (Lenz, 1988). If the inventory is raised, the manufacturing environment becomes that of the job shop type. On the contrary, if the operations are balanced, the environment becomes that of the transfer line. The changes in production are related to both inventory changes as well as changes in flow time. Three variables determine the amount of integration effects that result in a production process. These are inventory level, balanced loading, and flexibility. Inventory level is quantified by counting the number of parts that are active in the production process. Balanced loading can be quantified by the use of flow time. The use of flow time is to measure the balance within a production facility, and it is derived from the transfer line. The flow time provides a means to measure the balance between station loads in any type of production facility. Flexibility can be measured by the variability of the flow time. A process with greater degree of flexibility will provide less variability to the flow time.

Meredith and Suresh (1986) addressed justification of economic analysis and of analytical and strategic approaches in advanced manufacturing technologies. Evaluation of FMS alternatives was earlier carried out by Miltenburg and Krinsky (1987). They analyzed traditional economic evaluation techniques for the evaluation. Nelson (1986) formulated a scoring model for FMS project selection. Performance measures, viz., quality and flexibility, were also quantified in the scoring model. Use of the analytic hierarchy process (AHP) for evaluation of tangible and intangible benefits during FMS investment was reported by Wabalickis (1988). Stam and Kuula (1991) developed a two-phase decision support procedure using AHP and multi-objective mathematical programming for selection of FMS. Sambasivarao and Deshmukh (1997) presented a DSS integrating multi-attribute analysis, economic analysis and risk evaluation analysis. They have suggested AHP, TOPSIS (technique for order preference for similarity to ideal solution), and a linear additive utility model as an alternative multi-attribute analysis model. Shang and Sueyoshi (1995) formulated a model of simulation and data envelopment analysis (DEA) along with AHP for FMS selection. Karsak and Tolga (2001) proposed a fuzzy-MCDM approach for evaluation of advanced manufacturing system investments considering economic and strategic selection criteria. Karsak (2002) proposed a robust decision-making procedure for evaluating FMS using a distance-based fuzzy-MCDM philosophy.

Some researchers (Chen et al., 1998; Evans and Brown, 1989) believe that qualitative benefits cannot be considered mathematically unless one uses a knowledge-based system. This dissertation outlines a mathematical approach based on the judgmental values of a decision maker that can help decision makers in selecting the cost-effective FMS.

Abdel-Malek and Wolf (1991) propose a "measure" for the decision-making process. The said "measure" ranks different competing FMS designs according to their inherent flexibility as they relate to the maximum flexibility possible stipulated by the state-of-the-art. In developing the proposed "measure," the attributes governing the flexibility of FMS major components are defined. A notion of "strings" representing alternative production routes for different products is set forth. The method allows the integration of the eight points of flexibility stated by Browne et al. (1984) into a single comprehensive flexibility indicator.

Elango and Meinhart (1994) provide a framework for selection of an appropriate FMS using a holistic approach. The selection process considers operational and financial aspects. Furthermore, their selection process is consistent with industry, market, organizational, and other strategic needs.

A DSS for dynamic task allocation in a distributed structure for flexible manufacturing systems FMS has been developed by Trentesaux et al. (1998). An entity of the manufacturing system is considered as an autonomous agent, called the integrated management station (IMS), able to cooperate with other agents to achieve a global production program. Cooperation is performed by exchanging messages among the different agents. The characteristics of a DSS that supports multi-criteria algorithms and sensitivity tests is presented in Trentesaux et al. (1998). This DSS is integrated to each decision system of every IMS. Trentesaux et al.'s (1998) research work aims at allocating tasks in a dynamic way by proposing to the human operator a selection of possible resources.

Sarkis and Talluri (1999) disclose a model for evaluating alternative FMSs by considering both quantitative and qualitative factors. The evaluation process uses a DEA model, which incorporates both ordinal and cardinal measures. The model provides pair-wise comparisons of specific alternatives for FMSs. The consideration of both tangible and intangible factors is achieved in their methodology. The analysis of results provides both seller's and buyer's perspectives of FMS evaluation.

The decision-making process for machine-tool selection and operation allocation in a FMS usually involves multiple conflicting objectives. Rai et al.

(2002) address application of a fuzzy goal-programming concept to model the problem of machine-tool selection and operation allocation with explicit considerations given to objectives of minimizing the total cost of machining operation, material handling, and set up. The constraints pertaining to the capacity of machines, tool magazine, and tool life are included in the model. A genetic algorithm (GA)-based approach is adopted to optimize this fuzzy goal-programming model.

Advanced computing/communications technology is present in virtually all areas of manufacturing. In the near future, a totally computer-controlled manufacturing environmental will be a realistic expectation (Haddock and Hartshorn, 1989). The integration and enhancement of both computer-aided design (CAD) and computer-aided manufacturing (CAM) represents the foundations for achieving a totally integrated manufacturing system.

The requirements for increased responsiveness to market and the demands for shorter product introduction times underline the need for a coherent formal approach toward equipment selection to support the knowledge and experience of the engineers entrusted with this important task (Gindy and Ratchev, 1998). With the increasing complexity of the decision making in manufacturing system design, the search for the right structure depends on the capability of the designers to compare different solutions using common approaches in an integrated decision-making environment (Gindy and Ratchev, 1998).

Thus, machine tool selection has strategic implications that contribute to the manufacturing strategy of a manufacturing organization (Yurdakul, 2004). In such a case, it is important to identify and model the links between machine tool alternatives and manufacturing strategy (Yurdakul, 2004).

Haddock and Hartshorn (1989) present a DSS that assists in the specific selection of a machine required to process specific dimensions of a part. The selection will depend on part characteristics, which are labeled in a part code and correlated with machine specifications and qualifications. The choice of the optimal machine, versus possible alternates, is made by a planner comparing a criterion measure. Some possible criteria for selection as suggested by Haddock and Hartshorn (1989) are the relative location of machines, machining cost, processing time and availability of a machine.

Tabucanon et al. (1994) propose an approach to the design and development of an intelligent DSS that is intended to help the selection process of alternative machines for FMS. The process consists of a series of steps starting with an analysis of the information and culminating in a conclusion—a selection from several available alternatives and verification of the selected alternative to solve the problem. The approach combines the AHP technique with the rule-based technique for creating expert systems (ESs). This approach determines the architecture of the computer-based environment necessary for the decision support software system to be created. It includes the AHP software package (Expert Choice), Dbase III + DBMS, Expert System shell (EXSYS), and Turbo Pascal compiler (for the external procedural programs). A prototype DSS for a fixed domain, namely a CNC turning center that is required to process a family of rotational parts, is developed. Tabucanon et al.'s (1994) methodology helps the user to find the most "satisfactory" machine on the basis of several objective as well as subjective attributes.

Flexible manufacturing cells (FMCs) have been used as a tool to implement flexible manufacturing processes to increase the competitiveness of manufacturing systems (Wang et al., 2000). In implementing an FMC, decision makers encounter the machine selection problem, including attributes, e.g., machine type, cost, number of machines, floor space, and planned expenditures (Wang et al., 2000). Wang et al. (2000) propose a fuzzy multiple-attribute decision-making (FMADM) model to assist the decision maker to deal with the machine selection problem for an FMC realistically and economically. In their work, the membership functions of weights for those attributes are determined in accordance with their distinguishability and robustness when the ranking is performed.

AHP is widely used for tackling FMS selection problems due to the concept's simplicity and efficiency (Sambasivarao and Deshmukh, 1997). But AHP, as it is, do not take into consideration tangible factors, such as cost factors (Saaty, 1980, 1986, 1990). Thus, there is a need to allow cardinal factors in AHP to make the model robust and more efficient. In this chapter, a robust MCDM procedure is proposed using AHP that incorporates qualitative as well as quantitative measures for the FMS selection problem. The methodology proposed is very useful first to quantify the intangible factors in a strong manner and then to find out the best among member alternatives depending on their cost factors.

Some researchers believe that qualitative benefits cannot be considered mathematically unless one uses a knowledge-based system (Chen et al., 1998; Evans and Brown, 1989). This chapter outlines a fuzzified intelligent approach based on the judgmental values of the decision maker in selecting the most cost-effective FMS. One objective of this chapter is to find out fuzziness patterns of FMS selection decisions having a disparate level-of-satisfaction of the decision makers. Another objective is to provide a robust, quantified monitor of the level of satisfaction among decision makers and to calibrate these levels-of-satisfaction against decision makers' expectations.

2. FMS SELECTION PROBLEM

As a first step in testing the MCDM model proposed in the previous chapter, the authors have illustrated an example with FMS selection. Six different types of objective cost components have been identified for the selection problem. The total costs of each alternative are nothing, but the objective factor costs (OFCs) of the FMSs (refer to Table 1). The task is to select the best candidate-FMS among five candidate-FMSs.

Table 1. Cost Factor Components

FMS/OFCs	S_1	S_2	S_3	S_4	S_5
1. Cost of Acquisition	1.500	0.800	1.300	1.00	0.900
2. Cost of Installation	0.075	0.061	0.063	0.053	0.067
3. Cost of Commissioning	0.063	0.052	0.055	0.050	0.061
4. Cost of Training	0.041	0.043	0.046	0.042	0.040
5. Cost of Operation	0.500	0.405	0.420	0.470	0.430
6. Cost of Maintenance	0.500	0.405	0.420	0.470	0.430
Total Cost (OFC)	2.239	1.431	1.949	1.669	1.550
Objective Factor Measure (OFM_i)	0.154	0.241	0.177	0.206	0.222

The subjective attributes influencing the selection of FMS are shown in Table 2. The study consists of five different attributes, viz., flexibility in pick-up and delivery, flexibility in the conveying system, flexibility in automated storage and retrieval system, life expectancy/payback period, and tool magazine changing time. One may consider other attributes appropriate to selection of FMS. The attributes influencing the FMS selection problem are shown in Table 2.

Table 2. Attributes Influencing the FMS Selection Problem

Factor I	Factor II	Factor III	Factor IV	Factor V
Flexibility in pick-up and delivery	Flexibility in conveying system	Flexibility in automated storage and retrieval system	Life expectancy/pay back period	Tool magazine changing time

The MATLAB® fuzzy toolbox has been used in this work wherein a logical intelligent rule has been coded in M-file suitably using the designed MF.

3. SIMULATION USING MATLB®

The most important task for a decision maker is the selection of the factors. Thorough representation of the problem indicating the overall goal, criteria, sub-criteria (if any), and alternatives in all levels maintaining the sensitivity to change in the elements is a vital issue. The number of criteria or alternatives in the proposed methodology should be reasonably small to allow consistent pair-wise comparisons.

Matrix 1 is the decision matrix based on the judgmental values from different judges. Matrices 2 to 6 show comparisons of the weightages for each attribute. Matrix 7 consolidates the results of the earlier tables in arriving at the composite weights, i.e., SFM_i values, of each of the alternatives.

$$D = \begin{bmatrix} 1 & 5 & 3 & 4 & 5 \\ \frac{1}{5} & 1 & \frac{1}{3} & \frac{1}{2} & 1 \\ \frac{1}{3} & 3 & 1 & 3 & 5 \\ \frac{1}{4} & 2 & \frac{1}{3} & 1 & 3 \\ \frac{1}{5} & 1 & \frac{1}{5} & \frac{1}{3} & 1 \end{bmatrix}$$

Matrix 1. Decision matrix (I.R. = 4.39%)

$$A_1 = \begin{bmatrix} 1 & 3 & 2 & 5 & 4 \\ \frac{1}{3} & 1 & \frac{1}{3} & 5 & 2 \\ \frac{1}{2} & 3 & 1 & 4 & 3 \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{4} & 1 & \frac{1}{3} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{3} & 3 & 1 \end{bmatrix}$$

Matrix 2. Pair-wise comparison matrix for 'F_1' (I.R. = 4.48%)

$$A_2 = \begin{bmatrix} 1 & 7 & 3 & 5 & 6 \\ \frac{1}{7} & 1 & \frac{1}{4} & \frac{1}{3} & \frac{1}{2} \\ \frac{1}{3} & 4 & 1 & 3 & 4 \\ \frac{1}{5} & 3 & \frac{1}{3} & 1 & 2 \\ \frac{1}{6} & 2 & \frac{1}{4} & \frac{1}{2} & 1 \end{bmatrix}$$

Matrix 3. Pair-wise comparison matrix

$$A_3 = \begin{bmatrix} 1 & 4 & 1 & 3 & 7 \\ \frac{1}{4} & 1 & \frac{1}{4} & \frac{1}{2} & 5 \\ 1 & 4 & 1 & 2 & 7 \\ \frac{1}{3} & 2 & \frac{1}{2} & 1 & 3 \\ \frac{1}{7} & \frac{1}{5} & \frac{1}{7} & \frac{1}{3} & 1 \end{bmatrix}$$

Matrix 4. Pair-wise comparison matrix for F_2 (I.R. = 3.32%) for F_3 (I.R. = 1.88%)

$$A_4 = \begin{bmatrix} 1 & \frac{1}{3} & 5 & 3 & 6 \\ 3 & 1 & 5 & 7 & 6 \\ \frac{1}{5} & \frac{1}{5} & 1 & 2 & 3 \\ \frac{1}{3} & \frac{1}{7} & \frac{1}{2} & 1 & 2 \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{3} & \frac{1}{2} & 1 \end{bmatrix}$$

Matrix 5. Pair-wise comparison matrix

$$A_5 = \begin{bmatrix} 1 & \frac{1}{3} & 5 & 7 & 4 \\ 3 & 1 & 5 & 6 & 4 \\ \frac{1}{5} & \frac{1}{5} & 1 & 2 & \frac{1}{2} \\ \frac{1}{7} & \frac{1}{6} & \frac{1}{2} & 1 & \frac{1}{3} \\ \frac{1}{4} & \frac{1}{4} & 2 & 3 & 1 \end{bmatrix}$$

Matrix 6. Pair-wise comparison matrix for F_4 (I.R. = 6.22%) and for F_5 (I.R. = 6.87%)

$$G = \begin{bmatrix} 0.471 & 0.076 & 0.259 & 0.131 & 0.063 \\ 0.408 & 0.512 & 0.366 & 0.273 & 0.305 \\ 0.159 & 0.051 & 0.104 & 0.501 & 0.458 \\ 0.279 & 0.246 & 0.338 & 0.103 & 0.074 \\ 0.050 & 0.117 & 0.151 & 0.075 & 0.047 \\ 0.103 & 0.075 & 0.040 & 0.047 & 0.116 \end{bmatrix}$$

Matrix 7. Final matrix to find out Global Priority

In the proposed methodology, the unit of OFC is US$, whereas the objective factor measure (OFM) is a non dimensional quantity. Correspondingly, the SI is also a non-dimensional quantity. The higher the SI values, the better would be the selection. The value of the objective factor decision weight (α) lies between 0 and 1. For α = 0, SI = SFM; i.e., selection is solely dependent on subjective factor measure values found from AHP and SFM values dominate over OFM values. There is no significance of considering the cost factor components for α = 0. For α = 1, SI = OFM; i.e., OFM values dominate over the SFM values, and the FMS selection is dependent on OFM values only. For α = 1, the cost factors get priority over the other factors. Keeping this in mind, the values of α are taken in between 0 and 1. To verify the practicality and effectiveness of the final outcome of the proposed methodology, sensitivity analysis is done.

The basic fuzzified equation governing the selection process is recalled once again. It is to be remembered that the Eq. (1) (Wabalickis, 1988) uses MF as depicted by Eq. (2).

$$\left.\tilde{\mathrm{LSI}}_i\right|_{\alpha=\alpha_{\mathrm{SFM}_i}} = \mathrm{LSI}_L + \left(\frac{\mathrm{LSI}_U - \mathrm{LSI}_L}{\gamma}\right)\ln\frac{1}{C}\left(\frac{A}{\alpha_{\mathrm{LSI}_i}} - 1\right) \tag{1}$$

$$\mu(x) = \begin{cases} 1 & x < x^a \\ 0.999 & x = x^a \\ \dfrac{B}{1+Ce^{\gamma x}} & x^a < x < x^b \\ 0.001 & x = x^b \\ 0 & x > x^b \end{cases} \tag{2}$$

The intelligent decision algorithm generates the coefficients of the fuzzy constraints in the decision variables. The rule first declares a function C_j and assigns the constants in the MF. The aim is to produce a rule that works well on previously unseen data, i.e., the decision rule should "generalize" well. An example is appended below:

```
function [cj] = mpgen(cj0,cj1,gamma,mucj)
B = (0.998 / ((0.001 × exp(gamma)) − 0.999));
A=0.999 × (1 + B);
cj=cj0 + ((cj1 − cj0) / gamma) × (log((1 / B) × ((A / mucj) − 1)));
```

The rule supports this work by allowing the call to the function to contain a variable, which is automatically set to different values as one may request. The logical way in which the intelligent fuzzy-MCDM acts as an agent in the entire system includes many *if − else* rules.

3.1 Fuzzy Sensitivity of the MCDM Model

In a real-life situation, the decision environments rarely remain static. Therefore, it is essential to equip the proposed decision-making model with the capability to determine changes in the selection process that results from making changes in the parameters of the model. So, the dynamic behavior of the optimal selection found from the proposed methodology can be checked through the fuzzy-sensitivity plots.

Among all the FMSs, FMS_1 has the highest SI value when the objective factor decision weight lies between 0.33 and 1.00. However, FMS_2 would be preferred to other FMS candidate-alternatives when the value of level-of-satisfaction lies between 0.00 and 0.33.

The appropriate value of the level-of-satisfaction is to be selected cautiously. The reason behind this is as follows. The higher the α value, the dominance of the SFM_i values will be higher. The lower the α value, more will be the dominance of cost factor components, and subsequently, the intangible factors will get less priority.

Table 3 illustrates the final ranking based on the proposed model. From the Table 3 and Figures 16 to 20 ranking of the candidate-alternatives is $FMS_1 \succ FMS_2 \succ FMS_3 \succ FMS_5 \succ FMS_4$, i.e., FMS_1 is the best alternative at decision maker's level-of-satisfaction α = 0.42. Table 3 is a clear indication of accepting the proposed methodology for the selection problem in a conflicting-criteria environment.

Relationship between the degree of fuzziness, γ, versus level-of-satisfaction (α) has been depicted for all candidate-FMSs by Figures 1 to 5. This is a clear indication that the decision variables allow the MCDM model to achieve a higher level-of-satisfaction with a lesser degree of fuzziness. Figures 6 to 10 and 11 to 15 delineate SI indices versus level-of-satisfaction (α) and SI indices versus degree of fuzziness (γ), respectively.

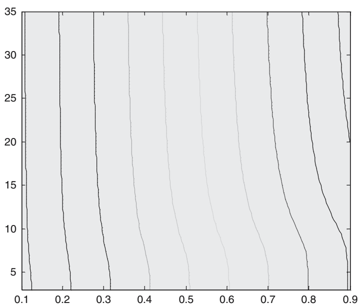

Figure 1. Fuzziness (γ) vs. α contour plot for FMS_1

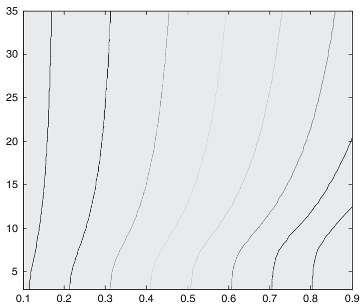

Figure 2. Fuzziness (γ) vs. α contour plot for FMS_2

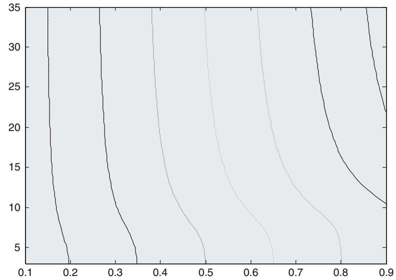

Figure 3. Fuzziness (γ) vs. α contour plot for FMS_3

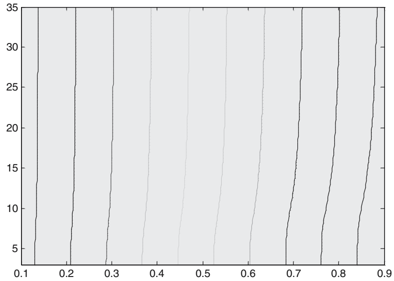

Figure 4. Fuzziness (γ) vs. α contour plot for FMS_4

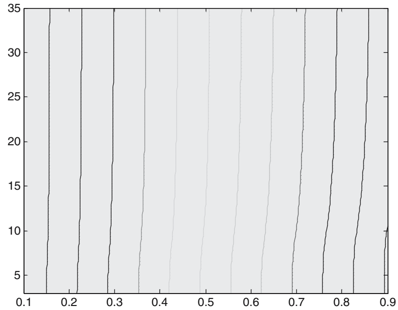

Figure 5. Fuzziness (γ) vs. α contour plot for FMS_5

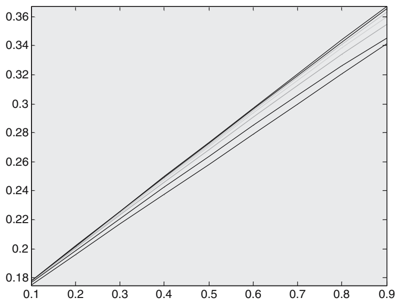

Figure 6. SI vs. α contour plot for FMS_1

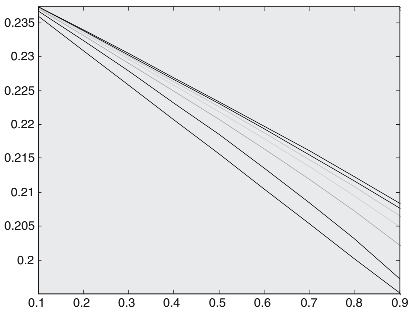

Figure 7. SI vs. α contour plot for FMS_2

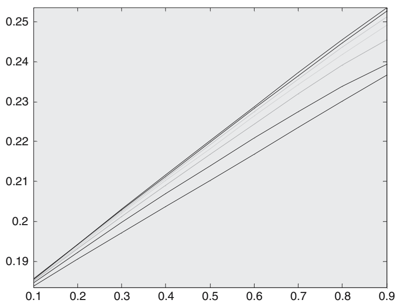

Figure 8. SI vs. α contour plot for FMS_3

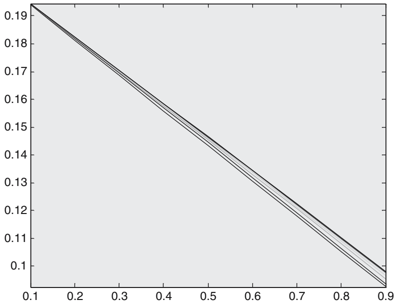

Figure 9. SI vs. α contour plot for FMS_4

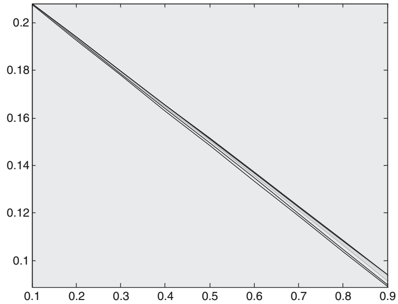

Figure 10. SI vs. α contour plot for FMS_5

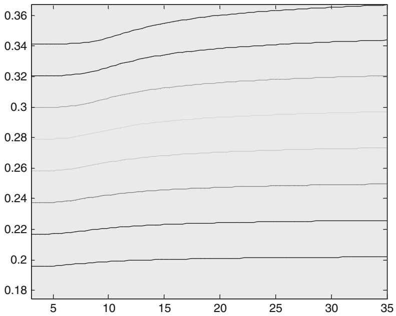

Figure 11. SI vs. γ contour plot for FMS_1

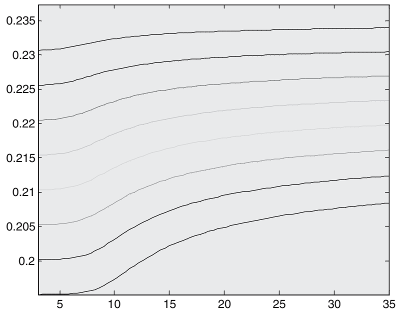

Figure 12. SI vs. γ contour plot for FMS_2

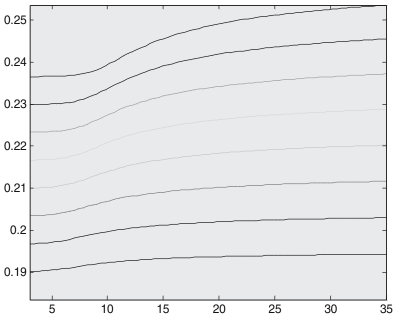

Figure 13. SI vs. γ contour plot for FMS_3

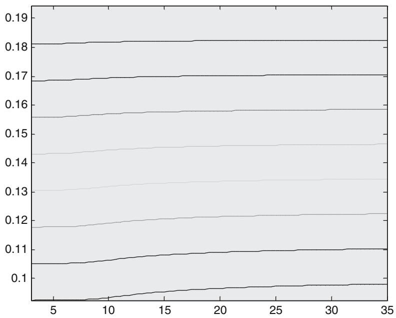

Figure 14. SI vs. γ contour plot for FMS_4

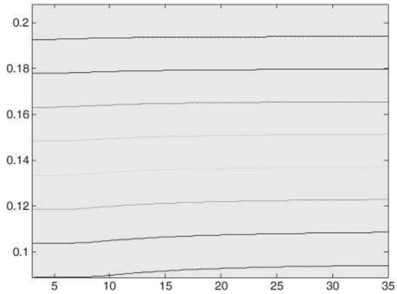

Figure 15. SI vs. γ contour plot for FMS_5

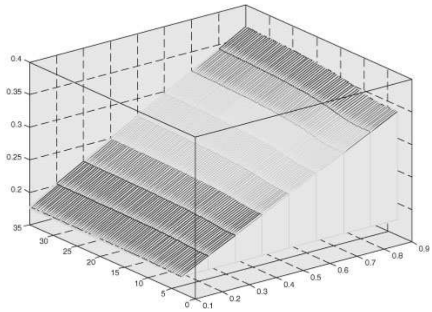

Figure 16. Fuzzy-sensitivity for FMS_1

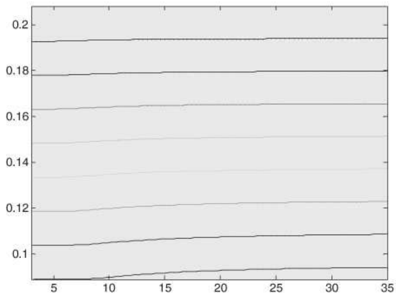

Figure 17. Fuzzy-sensitivity for FMS_2

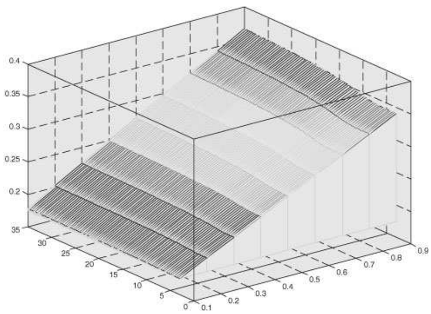

Figure 18. Fuzzy-sensitivity for FMS_3

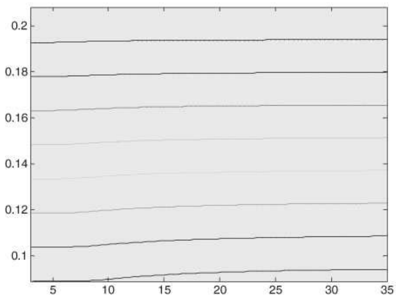

Figure 19. Fuzzy-sensitivity for FMS_4

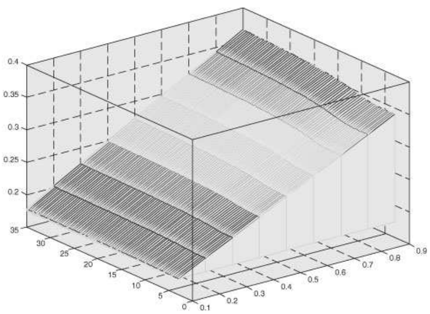

Figure 20. Fuzzy-sensitivity for FMS_5

Combining the plots as illustrated in Figures 1−15, one gets Figures 16−20. These figures elucidate 3-D mesh and contour plots. Basically Figures 16−20 illustrate fuzzy-sensitivity indicating relationships among SI indices, γ and α. Furthermore, from these plots, it is seen that the decision variables, as defined in Eq. (1), allow the MCDM model to achieve a higher level-of-satisfaction (α) with a lesser degree of fuzziness (γ).

Table 3. Ranking of the Systems

Candidate-FMS	SIi	Rank #
FMS1	0.249	#1
FMS2	0.224	#2
FMS3	0.210	#3
FMS4	0.155	#5
FMS5	0.162	#4

According to Table 3, the best alternative is FMS1 with the selection index of 0.249. The worst alternative is FMS4 with the selection index of 0.155.

4. GENERAL DISCUSSIONS AND CONCLUSION

This chapter outlined an intelligent fuzzy-MCDM model for appropriate selection of an FMS in a conflicting criteria environment. The proposed method calculates the GP for functional, design factors and other important attributes by eigenvector method of pair-wise comparison. These GPs are used as SFMs in determining SI.

In a real-life situation, the decision environments rarely remain static. So, the dynamic behavior of the optimal selection found from the proposed methodology has been checked through the fuzzy-sensitivity plots. Figures 16−20 teach an interesting phenomenon that is found in nature. At a lower level-of-satisfaction (α), the chances of getting involved in a higher degree of fuzziness (γ) increase. Therefore, a decision maker's level-of-satisfaction should be at least moderate in order to avoid higher degree of fuzziness while making any kind of decision using the proposed MCDM model delineated in the previous chapter.

The methodology proposed is very useful first in quantifying the intangible factors in a strong manner and then in finding out the best among the alternatives depending upon their cost factors. Contrary to the traditional way of selection using discounted cash flow (DCF), this methodology is a sound alternative to apply under an unstructured environment. The fuzzy-sensitivity strengthens the validity of the proposed methodology. It verifies the practicability as well as the effectiveness of the proposed DSS method.

It is not possible for an individual to consider all the factors related to FMS as follows:

- FMSs are available in a wide range,
- Performance standards of the systems are not uniform, and
- Expression of capabilities and performance attributes among manufacturers are inconsistent and incommensurable.

Thus, a decision-making expert system may help the decision maker in selecting the most cost-effective FMS considering the conflicting-in-nature factors of the systems.

The selection problem of FMS is complex due to the high capital costs involved and to the presence of multiple conflicting criteria. One can reduce investment and maintenance costs, increase equipment utilization, increase efficiency, as well as improve facilities layout by selecting the right system suitable for the operations to be carried out.

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SIMULATION SUPPORT TO GREY-RELATED ANALYSIS: DATA MINING SIMULATION

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Abstract: This chapter addresses the use of Monte Carlo simulation to reflect uncertainty as expressed by fuzzy input. Fuzziness is expressed through grey-related analysis, using interval fuzzy numbers. The method standardizes inputs through norms of interval number vectors. Interval-valued indexes are used to apply multiplicative operations over interval numbers. The method is demonstrated on a practical problem. Simulation offers a more complete understanding of the possible outcomes of alternatives as expressed by fuzzy numbers. The focus is on probability rather than on maximizing expected or extreme values.

Key words: Fuzzy sets, Monte Carlo simulation, grey-related analysis, data mining

1. INTRODUCTION

This chapter addresses the use of Monte Carlo simulation to reflect uncertainty as expressed by fuzzy input. Simulation offers a more complete understanding of the possible outcomes of alternatives as expressed by fuzzy numbers. The focus is on probability rather than on maximizing expected or extreme values. Both weights and alternative performance scores are allowed to be fuzzy. Both interval and trapezoidal fuzzy input can be considered (see Olson and Wu, 2005, 2006).

Fuzzy concepts have long been important in multiple criteria analysis (Dubois, 1980; Gau and Buehrer, 1993; Pawlak, 1982; Pearl, 1988; Pedrycz, 1998). Simulation has been applied to the analytical hierarchy

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process (AHP) (Levary and Wan, 1998), generating random pair-wise comparison input values. The uncertainty and fuzziness inherent in decision making makes the use of precise numbers problematic in multi-attribute models. Decision makers are usually more comfortable providing intervals for specific model input parameters. Interval input in multi-attribute decision making has been a very active field of research. Methods applying intervals have included (along with many others, see Zhang et al., 2005):

1. Use of interval numbers as the basis for ranking alternatives
   Brans and Vincke, 1985;
   El-Hawary, 1998;
   Chang and Yeh, 2004;
   Kahraman et al., 2004.
2. Error analysis with interval numbers
   Larichev and Moshkovich, 1991.
3. Use of linear programming and object programming with feasible regions bounded by interval numbers
   Roy, 1978;
   Liu et al., 1999;
   Royes et al., 2003.
4. Use of interval number ideal alternatives to rank alternatives by their nearness to the ideal
   Wang et al., 2004.

AHP was presented (Saaty, 1977) as a way to take subjective human inputs in a hierarchy and to convert these to a value function. This method has proven extremely popular. Salo and Hamalainen (1992) published their interval method using linear programming over the constrained space of weights and values as a means to incorporate uncertainty in decision-maker inputs to AHP hierarchies.

The problem of synthesizing ratio judgments in groups was considered very early in AHP (Aczel and Saaty, 1983). Fuzzy AHP was proposed as another way to reflect uncertainty in subjective inputs to AHP in the same group context (Buckley, 1984; 1985a; 1985b). Simulation has been presented as a way to rank order alternatives in the context of AHP values and weights (Levary and Wan, 1998).

Other multiple criteria methods besides AHP have considered fuzzy input parameters. ELECTRE (Roy, 1978) and PROMETHEE (Brans and Vincke, 1985) have always allowed fuzzy input for weights. A multi-attribute method involving fuzzy assessment for selection has been given in the airline safety domain (Chang and Yeh, 2004) and for multiple criteria selection of employees (Royes et al., 2003). Sensitivity in multi-attribute models with fuzzy inputs was considered by Aouam et al. (2003) and in goal programming by Fan et al. (2004). Rough set applications have also been presented (Zaras, 2004). This stream of research has obviously been rich and useful in application. It is extended by grey-related analysis.

2. GREY-RELATED ANALYSIS

Grey system theory was developed by Deng (1982) based on the concept that information is sometimes incomplete or unknown. The intent is the same as with factor analysis, cluster analysis, and discriminant analysis, except that those methods often do not work well when sample size is small and sample distribution is unknown (Wang et al., 2004). With grey-related analysis, interval numbers are standardized through norms, which allow transformation of index values through product operations. The method is simple, practical, and demands less-precise information than other methods. Grey-related analysis and TOPSIS (Hwang and Yoon, 1981; Lai et al., 1994; Yoon and Hwang, 1995) both use the idea of minimizing a distance function. However, grey-related analysis reflects a form of fuzzification of inputs and uses different calculations, to include a different calculation of norms. Feng and Wang (2001) applied grey relation analysis to select representative criteria among a large set of available choices and then used TOPSIS for outranking (Zhang et al., 2005)

Grey-related analysis has been used in a number of applications, In our discussion, we shall use the concept of the norm of an interval number column vector, the distance between intervals, product operations, and number-product operations of interval numbers.

Let $a = [a^-, a^+] = \{x \mid a^- \le x \le a^+, a^- \le a^+, a^-, a^+ \in R\}$.

We call $a = [a^-, a^+]$ an interval number. If $0 \le a^- \le a^+$, we call interval number $a = [a^-, a^+]$ a positive interval number.

Let $X = ([a_1^-, a_1^+], [a_2^-, a_2^+], \ldots, [a_n^-, a_n^+])^T$ be an n-dimension interval number column vector.

DEFINITION 1.

If $X = ([a_1^-, a_1^+], [a_2^-, a_2^+], ..., [a_n^-, a_n^+])^T$ is an arbitrary interval number column vector, the norm of X is defined here as

$$\| X \| = \max(\max(|a_1^-|, |a_1^+|), \max(|a_2^-|, |a_2^+|), ..., \max(|a_n^-|, |a_n^+|)) \quad (1)$$

DEFINITION 2.

If $a = [a^-, a^+]$ and $b = [b^-, b^+]$ are two arbitrary interval numbers, the distance from $a = [a^-, a^+]$ to $b = [b^-, b^+]$, is defined as

$$|a - b| = \max(|a^- - b^-|, |a^+ - b^+|) \quad (2)$$

DEFINITION 3.

If k is an arbitrary positive real number, and $a = [a^-, a^+]$ is an arbitrary interval number, then $k \cdot [a^-, a^+] = [ka^-, ka^+]$ will be called the number-product between k and $a = [a^-, a^+]$.

DEFINITION 4.

If $a = [a^-, a^+]$ is an arbitrary interval number, and $b = [b^-, b^+]$ are arbitrary interval numbers, we shall define the interval number product $[a^-, a^+] \cdot [b^-, b^+]$ as follows:

$$\text{when } b^+ > 0 \;\; [a^-, a^+] \cdot [b^-, b^+] = [a^- b^-, a^+ b^+] \quad (3)$$

$$\text{when } b^+ < 0 \;\; [a^-, a^+] \cdot [b^-, b^+] = [a^+ b^-, a^- b^+] \quad (4)$$

If $b^+ = 0$, the interval reverts to a point, and thus, we would return to the basic crisp model.

2.1 Steps of Grey-Related Analysis

The principle and steps of the Grey-related analysis method are as follows:

Step 1. Construct decision matrix A with an index number of interval numbers. If the index value of the *j*th index G_j of feasible plan X_i is an interval number $[a_{ij}^-, a_{ij}^+]$, $i = 1,2,...,m$, $j = 1,2,...,n$, decision matrix A with index number of interval numbers is defined as the follows:

$$A = \begin{bmatrix} [a_{11}^-, a_{11}^+] & [a_{12}^-, a_{12}^+] & \dots & [a_{1n}^-, a_{1n}^+] \\ [a_{21}^-, a_{21}^+] & [a_{22}^-, a_{22}^+] & \dots & [a_{2n}^-, a_{2n}^+] \\ \dots & \dots & \dots & \dots \\ [a_{m1}^-, a_{m1}^+] & [a_{m2}^-, a_{m2}^+] & \dots & [a_{mn}^-, a_{mn}^+] \end{bmatrix} \quad (5)$$

Step 2. Transform the "contrary index" into a positive index .The index is called a positive index if a greater index value is better. The index is called a contrary index if a smaller index value is better. We may transform a contrary index into a positive index if the *j*th index G_j is a contrary index

$$[b_{ij}^-, b_{ij}^+] = [-a_{ij}^+, -a_{ij}^-] \quad i = 1,2,...,m\,. \quad (6)$$

Without loss of generality, in the following discussion, we supposed that all the indexes are "positive indices."

Step 3. Standardize decision matrix A with an index number of interval numbers, obtaining standardizing decision matrix $R = [\, r_{ij}^-, r_{ij}^+ \,]$. If we mark the column vectors of decision matrix A with interval-valued indexes with $A_1, A_2, ..., A_n$, the element of standardizing decision matrix $R = [\, r_{ij}^-, r_{ij}^+ \,]$ is defined as

$$[r_{ij}^-, r_{ij}^+] = \frac{[a_{ij}^-, a_{ij}^+]}{\| A_j \|}, \quad i = 1,2,...,m \qquad j = 1,2,...,n\,. \quad (7)$$

Step 4. Calculate interval number weighted matrix $C = ([\, c_{ij}^-, c_{ij}^+ \,])_{m \times n}$. The formula for the element of interval number weighted matrix C is $C = ([\, c_{ij}^-, c_{ij}^+ \,])_{m \times n}$
where

$$[c_{ij}^-, c_{ij}^+] = [c_j, d_j] \cdot [r_{ij}^-, r_{ij}^+], \quad i = 1,2,...,m \qquad j = 1,2,...,n\,. \quad (8)$$

Step 5. Determine reference number sequence. The element of reference number sequence is composed of the optimal weighted interval number index value for every alternative.

$U_0 = ([u_0^-(1)\ ,\ u_0^+(1)]\ ,\ [u_0^-(2)\ ,\ u_0^+(2)]\ ,\ \dots\ ,\ [u_0^-(n)\ ,\ u_0^+(n)])$

is a reference number sequence if $u_0^-(j) = \max_{1 \le i \le m} c_{ij}^-$, $u_0^+(j) = \max_{1 \le i \le m} c_{ij}^+$, $j = 1, 2, \dots, n$.

Step 6. Calculate connections between alternatives. First, calculate the connection coefficient $\xi_i(k)$ between the sequence composed of weight interval number standardized index values for every alternative $U_i = ([c_{i1}^-\ ,\ c_{i1}^+], [c_{i2}^-\ ,\ c_{i2}^+],\ \dots\ ,\ [c_{in}^-\ ,\ c_{in}^+])$ and the reference number sequence $U_0 = ([u_0^-(1),\ u_0^+(1)],\ [u_0^-(2),\ u_0^+(2)],\ \dots,\ [u_0^-(n),\ u_0^+(n)])$.

The formula for $\xi_i(k)$ is

$$\xi_i(k) = \frac{\min_i \min_k |[u_0^-(k), u_0^+(k)] - [c_{ik}^-, c_{ik}^+]| + \rho \max_i \max_k |[u_0^-(k), u_0^+(k)] - [c_{ik}^-, c_{ik}^+]|}{|[u_0^-(k), u_0^+(k)] - [c_{ik}^-, c_{ik}^+]| + \rho \max_i \max_k |[u_0^-(k), u_0^+(k)] - [c_{ik}^-, c_{ik}^+]|} \qquad (9)$$

Here, $\rho \in (0, +\infty)$, and ρ is a resolving coefficient. The smaller ρ is, the greater its resolving power. In general, $\rho \in [0,\ 1]$ .The value of ρ may be changed to reflect the desired degree of resolution.

After calculating $\xi_i(k)$, the connection between the *i*-th plan and the reference number sequence is calculated by the following formula:

$$r_i = \frac{1}{n} \times \sum_{k=1}^{n} \xi_i(k), \qquad i = 1, 2, \dots m \qquad (10)$$

Step 7. Determine optimal plan. The feasible plan X_t is optimal if $r_t = \max_{1 \le i \le m} r_i$.

3. MONTE CARLO SIMULATION

Fuzzy inputs can easily be simulated using Monte Carlo simulation models. Interval random numbers over the interval 0−1 can be generated in Monte Carlo simulation directly, and these can be converted to any other uniform range. Simulations can be easier to analyze if they are controlled, using unique seed values to ensure that the difference in simulation output due to random variation was the same for each alternative.

3.1 Trapezoidal Distributed Fuzzy Numbers

The trapezoidal fuzzy input dataset can also be simulated.

X is random number (0 < *rn* < 1).

Definition of trapezoida1 is left 0 in Figure 1; *a*2 is left 1; *a*3 is right 1; and *a*4 is right 0.

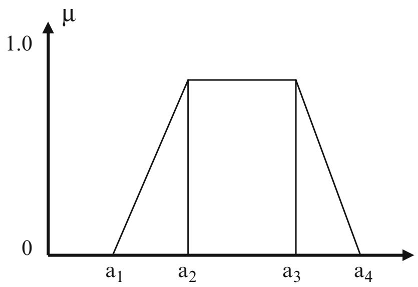

Figure 1. A trapezoidal fuzzy number

J is area of left triangle contingent calculation:

K is area of rectangle

L is area of right triangle

Fuzzy sum = left triangle + rectangle + right triangle = 1

M is the area of the left triangle plus the rectangle (for calculation of *X* value)

X is the random number drawn (which is the area)

If $X \le J$:

$$X = a1 + \sqrt{\frac{X \times (a2 - a1) \times (a4 - a3 + a2 - a1)}{J + L}} \tag{11}$$

If $J \le X \le J+K$:

$$X = a2 + \frac{X - J}{K} \times (a3 - a2) \tag{12}$$

If $J+K \le X$:

$$X = a4 - \sqrt{\frac{(1 - X) \times (a4 - a3) \times (a4 - a3 + a2 - a1)}{J + L}} \tag{13}$$

Our calculation is based on drawing a random number reflecting the area (starting on the left (a1) as 0, ending on the right (a4) as 1), and calculating the distance on the *X*-axis. The simulation software Crystal Ball was used to replicate each model 1000 times for each random number seed. The software enabled counting the number of times each alternative won.

3.2 Grey-Related Decision Tree Models

Grey-related analysis is expected to provide improvement over crisp models by better reflecting the uncertainty inherent in many human analysts' minds. Data mining models based on such data are expected to be less accurate, but hopefully not by very much (Hu et al., 2003). However, grey-related model input would be expected to be more stable under conditions of uncertainty where the degree of change in input data increased.

We applied decision tree analysis to a small set (1000 observations total) of credit card data. Originally, there was one output variable (whether or not the account defaulted, a binary variable with 1 representing default, 0 representing no default) and 65 available explanatory variables. These variables were analyzed, and 26 were selected as representing ideas that might be important to predicting the outcome. The original data set was imbalanced, with 140 default cases and 860 not defaulting. Initial decision tree models were almost all degenerate, classifying all cases as not defaulting. When differential costs were applied, the reverse degenerate model was obtained (all cases predicted to default). Therefore, a new dataset containing all 140 default cases and 160 randomly selected not default cases was generated, where 200 cases were randomly selected as a training set, with the remaining 100 cases used as a test set.

The explanatory variables included five binary variables and one categorical variable, with the remaining 20 being continuous. To reflect fuzzy input, each variable (except for binary variables) was categorized into three categories based on analysis of the data, using natural cutoff points to divide each variable into roughly equal groups.

Decision tree models were generated using the data mining software PolyAnalyst. That software allows setting minimum support level (the number of cases necessary to retain a branch on the decision tree), and a slider setting to optimistically or pessimistically split criteria. Lower support levels allow more branches, as does the optimistic setting. Every time the model was run, a different decision tree was able to be obtained.

But nine settings were applied, yielding many overlapping models. Three unique decision trees were obtained, which are reflected in the output to follow. A total of eight explanatory variables were used in these three decision trees. The same runs were made for the categorical data reflecting grey-related input. Four unique decision trees were obtained, with formulas again given below. A total of seven explanatory variables were used in these four categorical decision trees. All seven models and their fit on test data are given in the Appendix.

These models were then entered into a Monte Carlo simulation (supported by Crystal Ball software). A perturbation of each input variable was generated, set at five different levels of perturbation. The intent was to measure the loss of accuracy for crisp and grey-related models.

The model results are given in the seven model reports in the appendix. Since different variables were included in different models, it is not possible to directly compare relative accuracy as measured by fitting test data. However, the means for the accuracy on test data for each model given in Table 1 show that the crisp models declined in accuracy more than the categorical models. The column headings in Table 1 reflect the degree of perturbation simulated.

Table 1. Mean Model Accuracy

Model	**Crisp**	**0.25**	**0.50**	**1.00**	**2.00**	**3.00**	**4.00**
Cont. 1	0.70	0.70	0.70	0.68	0.67	0.66	0.65
Cont. 2	0.67	0.67	0.67	0.67	0.67	0.66	0.66
Cont. 3	0.71	0.71	0.70	0.69	0.67	0.67	0.66
Cont.	**0.693**	**0.693**	**0.690**	**0.680**	**0.670**	**0.667**	**0.657**
Cat. 1	0.70	0.70	0.68	0.67	0.66	0.66	0.65
Cat. 2	0.70	0.70	0.70	0.69	0.68	0.67	0.67
Cat. 3	0.70	0.70	0.70	0.69	0.69	0.68	0.67
Cat. 4	0.70	0.70	0.70	0.69	0.68	0.67	0.67
Cat.	**0.700**	**0.700**	**0.695**	**0.688**	**0.678**	**0.670**	**0.665**

The fuzzy models were expected to be less accurate, but here they actually average slightly better accuracy. This, however, can simply be attributed to different variables being used in each model. The one exception is that models Continuous 2 and Categorical 3 were based on one variable, V64, the balance-to-payment ratio. The cutoff generated by model Continuous 2 was 6.44 (if V64 was < 6.44, prediction 0), whereas the cutoff for Categorical 3 was 4.836 (if V64 was > 4.835, the category was "high," and the decision tree model was that if V64 = "high," prediction 1, else prediction 0). The fuzzy model here was actually better in fitting the test data (although slightly worse in fitting the training data).

The important point of the numbers in Table 1 is that there clearly was greater degradation in model accuracy for the continuous models than for the categorical (grey-related) models. This point is demonstrated further by the wider dispersion of the graphs in the Appendix.

4. CONCLUSIONS

This chapter has discussed the integration of grey-related analysis and decision making with uncertainty through simulation. Simulation provides a means to better visualize model results and a flexible way to include any level of uncertainty and complexity. Results based on Monte Carlo simulation as a data-mining technique offer more insights to assist our decision making in fuzzy environments by incorporating probability interpretation. Analysis of decision tree models through simulation shows that there does appear to be less degradation in model fit for grey-related (categorical) data than for decision tree models generated from raw continuous data. It must be admitted that this is a preliminary result, based on a relatively small dataset of only one type of data. However, it is intended to demonstrate a point meriting future research. This decision-making approach can be applied to large-scale datasets, expanding our ability to implement data mining and large-scale computing.

The easiest way to apply fuzzy concepts to data mining is to categorize data. This creates the problem of where to set limits between categories. However, reliance on expert judgment can often provide useful limits. If data-mining data are represented through fuzzy concepts, simulation can be applied. Since fuzzy data are probabilistic, simulation seems appropriate. Simulation does involve a lot more work than closed-form (crisp) datasets. However, fuzzy data are often a better representation of real domains.

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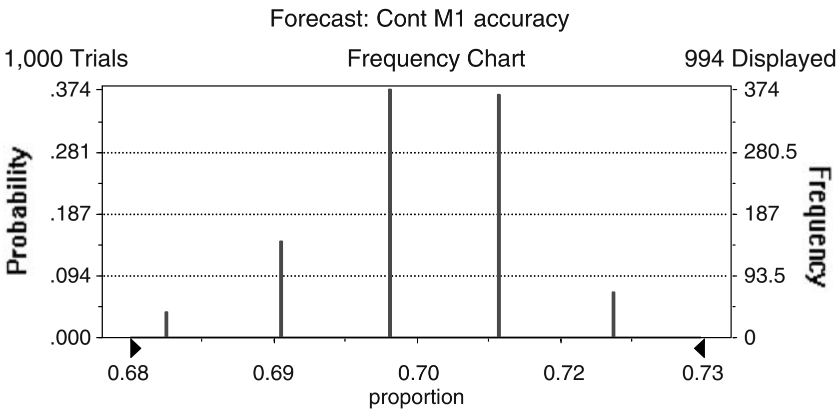
APPENDIX: MODELS AND THEIR RESULTS

Continuous Model 1:

IF(Bal/Pay<6.44,N,IF(Utilization<1.54,Y,IF(AvgPay<3.91,N,Y)))

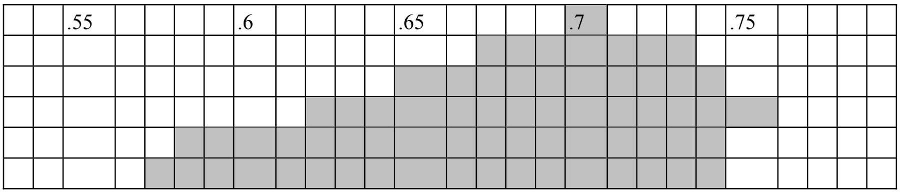


Test matrix:

	Model 0	Model 1	Accuracy
Actual 0	43	16	
Actual 1	14	27	0.70

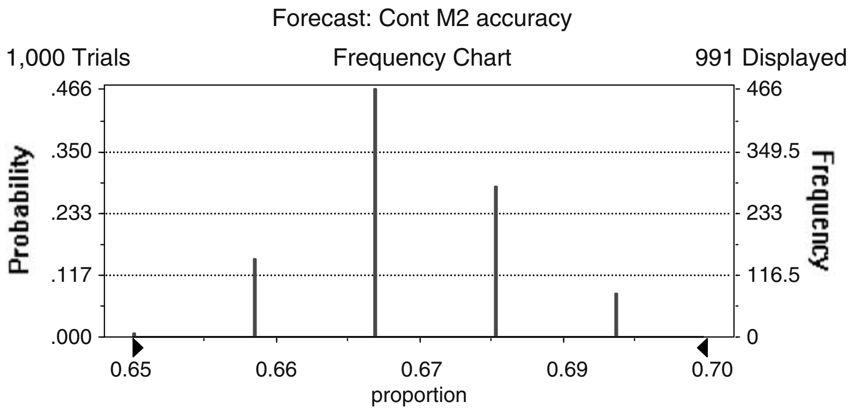
Simulation accuracy of 100 observations, 1000 simulation runs

perturbation	[−0.25,0.25]	0.67−0.73
perturbation	[−0.50,0.50]	0.65−0.74
perturbation	[−1,1]	0.62−0.75
perturbation	[−2,2]	0.58−0.74
perturbation	[−3,3]	0.57−0.74



Continuous Model 2:

IF (Bal/Pay<6.44,N,Y)

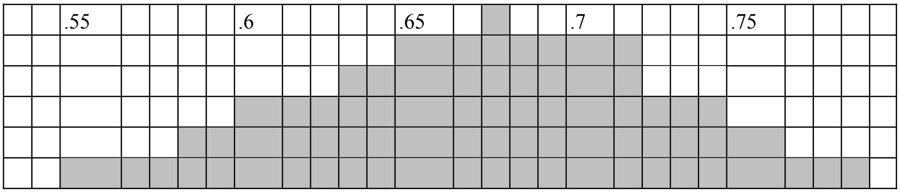


Test matrix:

	Model 0	Model 1	Accuracy
Actual 0	40	19	
Actual 1	14	27	0.67

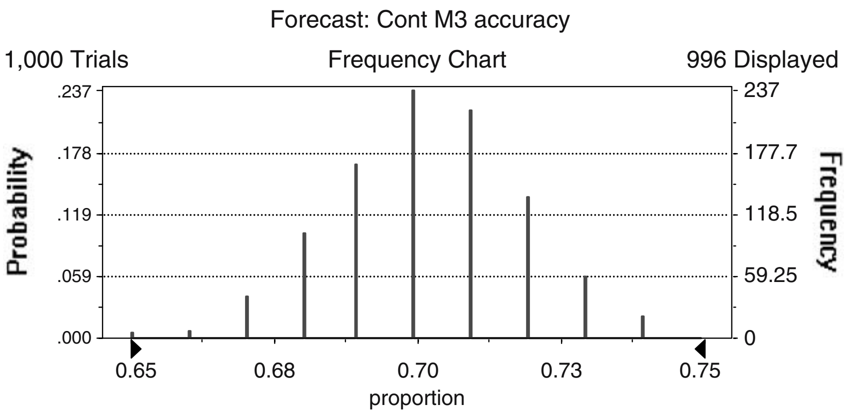
Simulation accuracy of 100 observations, 1000 simulation runs

perturbation [−0.25,0.25]	0.65−0.71
perturbation [−0.50,0.50]	0.63−0.71
perturbation [−1,1]	0.60−0.74
perturbation [−2,2]	0.58−0.75
perturbation [−3,3]	0.55−0.78



Continuous Model 3:

IF(Bal/Pay<6.44,N,IF(Utilization<1.54,Y,IF(AvgRevPay<2.28,Y,N)))



Test matrix:

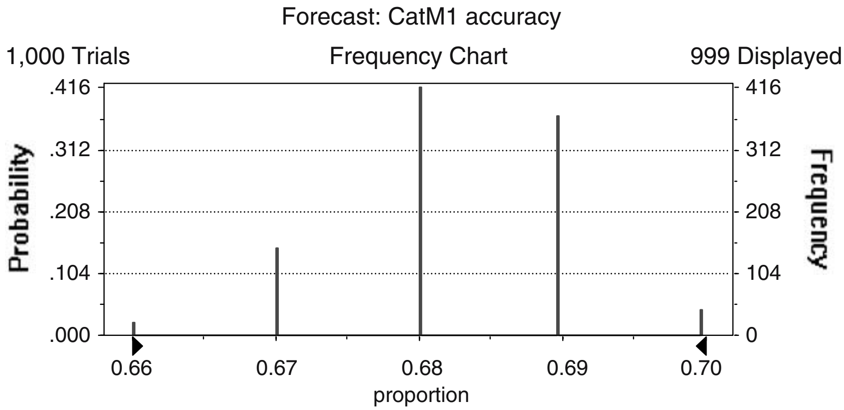
	Model 0	Model 1	Accuracy
Actual 0	44	15	
Actual 1	14	27	0.71

Simulation accuracy of 100 observations, 1000 simulation runs
perturbation [−0.25,0.25] 0.65−0.76
perturbation [−0.50,0.50] 0.63−0.76
perturbation [−1,1] 0.59−0.77
perturbation [−2,2] 0.54−0.79
perturbation [−3,3] 0.53−0.78

		.55					.6					.65					.7					.75				

Categorical Model 1:

IF(Bal/Pay<6.44,N,IF(Utilization<1.54,Y,IF(AvgRevPay<2.28,Y,N)))



Test matrix:

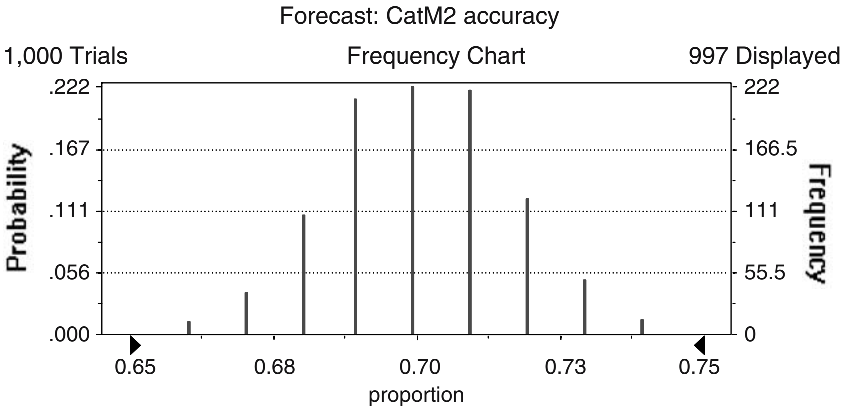
	Model 0	Model 1	Accuracy
Actual 0	33	26	
Actual 1	5	36	0.70

Simulation accuracy of 100 observations, 1000 simulation runs
perturbation [−0.25,0.25] 0.66−0.71
perturbation [−0.50,0.50] 0.64−0.71
perturbation [−1,1] 0.61−0.71
perturbation [−2,2] 0.58−0.73
perturbation [−3,3] 0.56−0.74

		.55					.6					.65					.7					.75				

Categorical Model 2:

IF(Bal/Pay=`”high”,IF(CredLine=”low”,
IF(CDL=”mid”,IF(Pur%Bal=”low”,Y,N),
IF(CDL=”low”,N,Y))
IF(CredLine=”high”,IF(CalcIntRate=”mid”,N,Y),Y),N)

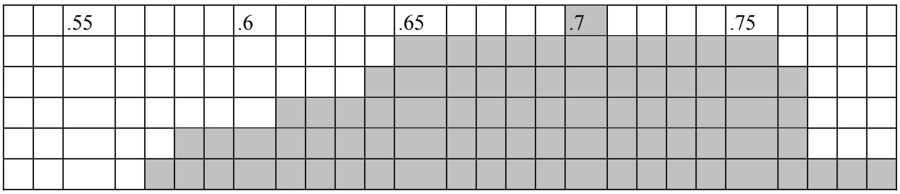


Test matrix:

	Model 0	Model 1	Accuracy
Actual 0	42	17	
Actual 1	13	28	0.70

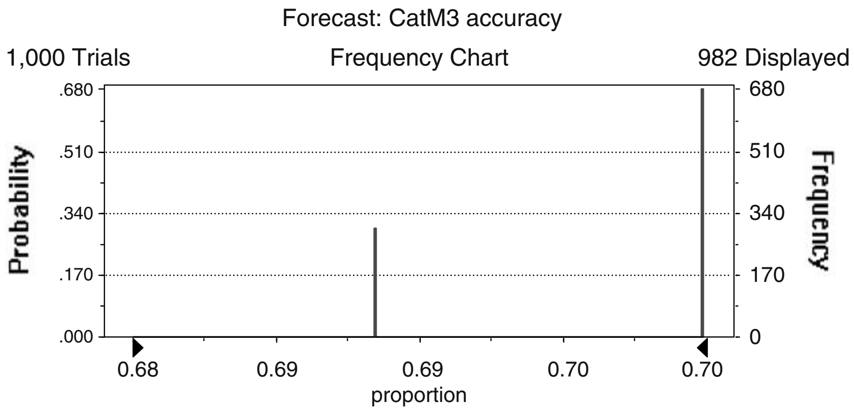
Simulation accuracy of 100 observations, 1000 simulation runs

perturbation [−0.25,0.25]	0.65−0.75
perturbation [−0.50,0.50]	0.64−0.76
perturbation [−1,1]	0.61−0.76
perturbation [−2,2]	0.58−0.76
perturbation [−3,3]	0.57−0.80



Categorical Model 3:

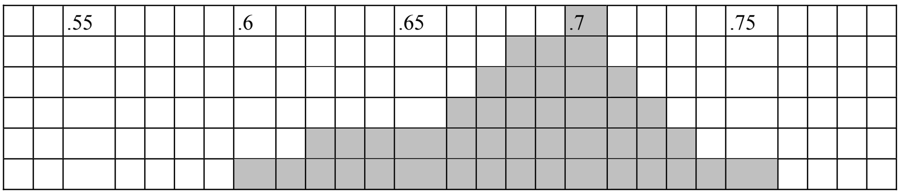
IF(Bal/Pay=”high”,Y,N)



Test matrix:

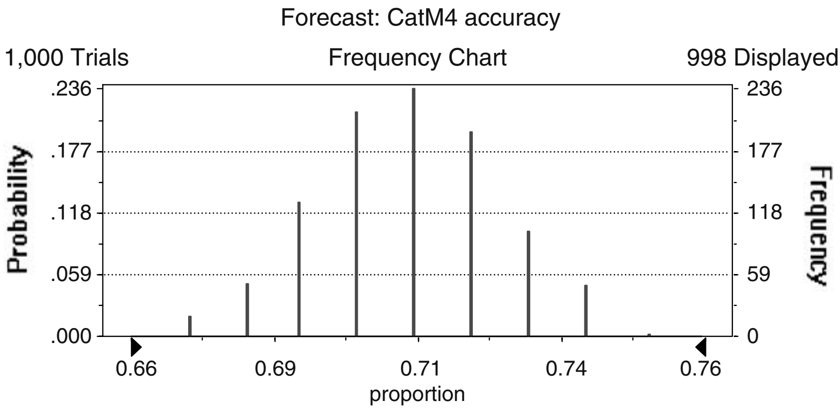
	Model 0	Model 1	Accuracy
Actual 0	33	26	
Actual 1	4	37	0.70

Simulation accuracy of 100 observations, 1000 simulation runs
perturbation [−0.25,0.25] 0.68−0.70
perturbation [−0.50,0.50] 0.67−0.71
perturbation [−1,1] 0.66−0.72
perturbation [−2,2] 0.62−0.73
perturbation [−3,3] 0.59−0.75



Categorical Model 4:

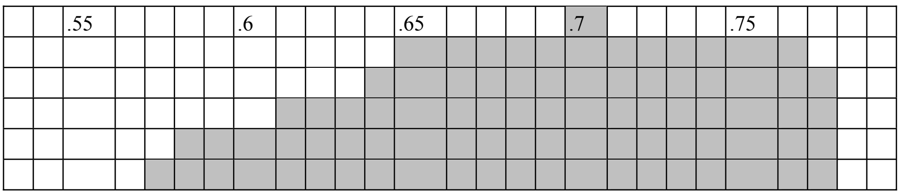
IF(Bal/Pay=”high”,
IF(CredLine=”low”,
IF(CDL=”mid”,IF(Purch%Bal=”low”,Y,N),
IF(CDL=”low”,IF(Residence<.5,Y,N),Y))
IF(CredLine=”high”,IF(CalcIntRate=”mid”,N,Y),Y)



Test matrix:

	Model 0	Model 1	Accuracy
Actual 0	41	18	
Actual 1	12	29	0.70

Simulation accuracy of 100 observations, 1000 simulation runs
perturbation [−0.25,0.25] 0.65−0.76
perturbation [−0.50,0.50] 0.64−0.77
perturbation [−1,1] 0.61−0.77
perturbation [−2,2] 0.58−0.77
perturbation [−3,3] 0.57−0.77



NEURO-FUZZY APPROXIMATION OF MULTI-CRITERIA DECISION-MAKING QFD METHODOLOGY

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Abstract: This chapter demonstrates how a neuro-fuzzy approach could produce outputs of a further-modified multi-criteria decision-making (MCDM) quality function deployment (QFD) model within the required error rate. The improved fuzzified MCDM model uses the modified S-curve membership function (MF) as stated in an earlier chapter. The smooth and flexible logistic membership function (MF) finds out fuzziness patterns in disparate level-of-satisfaction for the integrated analytic hierarchy process (AHP-QFD model. The key objective of this chapter is to guide decision makers in finding out the best candidate-alternative robot with a higher degree of satisfaction and with a lesser degree of fuzziness.

Key words: ANFIS, AHP, QFD, fuzziness patterns, decision-making, level-of-satisfaction

1. INTRODUCTION

Arriving at the decision to install a robot in a manufacturing firm can be a difficult and complicated process. Even after the initial decision to acquire a robot is made, the problem of which robot to select from the many that are available can confound managers who often lack the time and expertise to perform an extensive search and analysis. Furthermore, the current trend indicates that the number of robot manufacturers and suppliers are increasing as engineers continue to find more applications for robots. The

C. Kahraman (ed.), Fuzzy Multi-Criteria Decision Making.

problem of robot selection has become more difficult in recent years due to increasing complexity, available features, and facilities offered by different robotic products.

1.1 Concepts on Neuro-Fuzzy Systems

A fuzzy inference system (FIS) can use human expertise by storing its essential components in the rule base and the database and can perform fuzzy reasoning to infer the overall output value. The derivation of *if−then* rules and corresponding membership functions (MFs) depends heavily on the a priori knowledge about the system under consideration. However, there is no systematic way to transform experiences of knowledge of human experts into the knowledge base of an FIS. There is also a need for adaptability or some learning algorithms to produce outputs within the required error rate. On the other hand, ANN learning mechanism does not rely on human expertise. Due to the homogenous structure of ANN, it is hard to extract structured knowledge from either the weights or the configuration of the an artificial neural network (ANN). The weights of the ANN represent the coefficients of the hyperplane that partition the input space into two regions with different output values. If we can visualize this hyperplane structure from the training data, then the subsequent learning procedures in an ANN can be reduced. However, in reality, the a priori knowledge is usually obtained from human experts; it is most appropriate to express the knowledge as a set of fuzzy *if−then* rules, and it is not possible to encode into an ANN 0. Table 1 summarizes the comparison of FIS and ANN.

Table 1. Complementary Features of ANN and FIS

ANN	FIS
Black box	Interpretable
Learning from scratch	Making use of linguistic knowledge

To a large extent, the drawbacks pertaining to these two approaches seem complementary. Therefore it is natural to consider building an integrated system combining the concepts of FIS and ANN modeling. A common way to apply a learning algorithm to a FIS is to represent it in a special ANN−like architecture 0. However, the conventional ANN learning algorithms (gradient descent) cannot be applied directly to such a system as the functions used in the inference process are usually nondifferentiable. This problem can be tackled by using differentiable functions in the inference system or by not using the standard neural learning algorithm. In our simulation, we used an adaptive network based fuzzy inference system (ANFIS) (Jang, 1991).

ANFIS implements a Takagi Sugeno Kang (TSK) fuzzy inference system (Jang, 1991) in which the conclusion of a fuzzy rule is constituted by a weighted linear combination of the crisp inputs rather than by a fuzzy set.

For a first-order TSK model, a common rule set with two fuzzy *if−then* rules is represented as follows:

Rule 1: If x is A_1 and y is B_1, then $f_1 = p_1x + q_1y + r_1$

Rule 2: If x is A_2 and y is B_2, then $f_2 = p_2x + q_2y + r_2$

where x and y are linguistic variables and A_1, A_2, B_1, and B_2 are corresponding fuzzy sets and p_1, q_1, r_1 and p_2, q_2, r_2 are linear parameters.

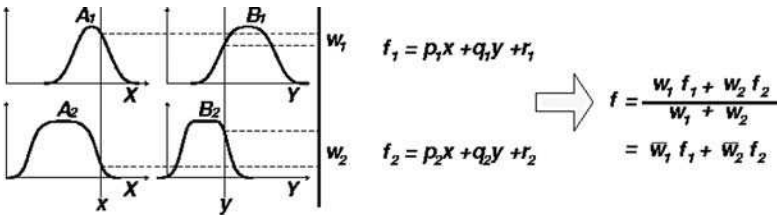

Figure 1. TSK−type fuzzy inference system

Figure 1 illustrates the TSK fuzzy inference system when two membership functions each are assigned to the two inputs (x and y). The TSK fuzzy controller usually needs a smaller number of rules, because their output is already a linear function of the inputs rather than a constant fuzzy set.

Figure 2 depicts the five-layered architecture of ANFIS, and the functionality of each layer is as follows:

Layer-1. Every node in this layer has a node function

$$O_i^1 = \mu_{Ai}(x), \text{ for i = 1, or 2}$$

$$O_i^1 = \mu_{B_{i-2}}(y), \text{ for i = 3,4,….}$$

O_i^1 is the membership grade of a fuzzy set A ($= A_1, A_2, B_1$ or B_2), and it specifies the degree to which the given input x (or y) satisfies the quantifier A. Usually the node function can be any parameterized function.

A Gaussian membership function is specified by two parameters c (membership function center) and σ (membership function width).

2. ADAPTIVE NETWORK-BASED FUZZY INFERENCE SYSTEM (ANFIS)

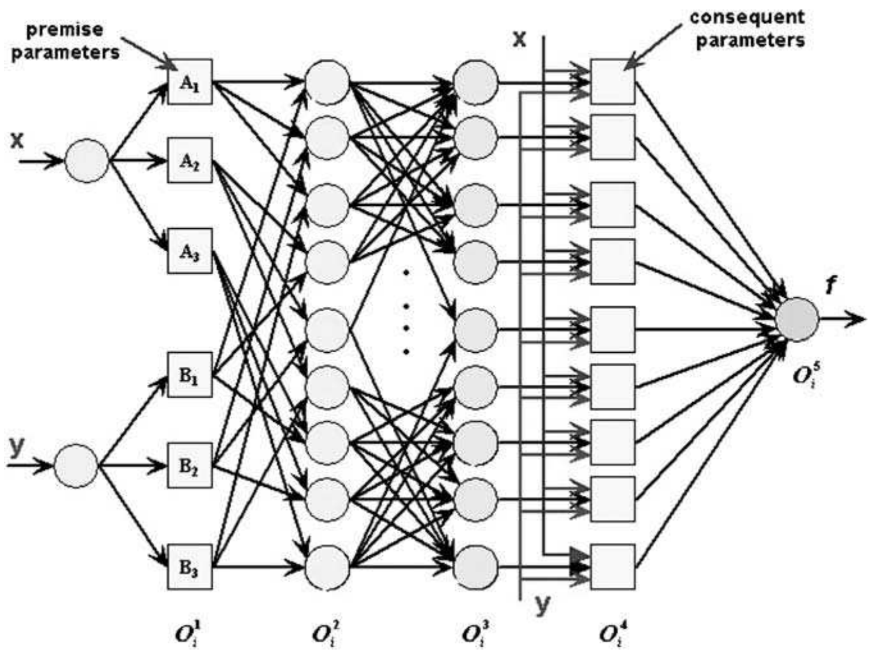

Figure 2. Architecture of the ANFIS

$$\text{Guassian}\,(x, c, \sigma) = e^{-\frac{1}{2}\left(\frac{x-c}{\sigma}\right)^2}.$$

Parameters in this layer are referred to as premise parameters.

Layer-2. Every node in this layer multiplies the incoming signals and sends the product out. Each node output represents the firing strength of a rule.

$$O_i^2 = w_i = \mu_{Ai}(x) \times \mu_{Bi}(y), i = 1,2 .$$

In general any T-norm operators perform fuzzy AND can be used as the node function in this layer.

Layer-3. Every *i*th node in this layer calculates the ratio of the *i*th rule's firing strength to the sum of all rule's firing strength.

$$O_i^3 = \overline{w_i} = \frac{w_i}{w_1 + w_2}, i = 1, 2 .$$

Layer-4. Every node *i* in this layer is with a node function

$$O_1^4 = \overline{w_i} f_i = \overline{w_i}(p_i x + q_i y + r_i) ,$$

where $\overline{w_i}$ is the output of layer-3, and $\{p_i, q_i, r_i\}$ is the parameter set. Parameters in this layer will be referred to as consequent parameters.

Layer-5. The single node in this layer computes the overall output as the summation of all incoming signals:

$$O_1^5 = \text{Overall output} = \sum_i \overline{w_i} f_i = \frac{\sum_i w_i f_i}{\sum_i w_i}$$

ANFIS makes use of a mixture of backpropagation to learn the premise parameters and least mean square estimation to determine the consequent parameters. A step in the learning procedure has two parts: In the first part, the input patterns are propagated, and the optimal conclusion parameters are estimated by an iterative least mean square procedure, whereas the antecedent parameters (membership functions) are assumed to be fixed for the current cycle through the training set. In the second part, the patterns are propagated again, and in this epoch, backpropagation is used to modify the antecedent parameters, whereas the conclusion parameters remain fixed. This procedure is then iterated (Jang, 1991).

3. QFD PROCESS

QFD is a method for structured product planning and development. It enables a development team to specify clearly the customer's requirement. It also evaluates each proposed product systematically in terms of its impact on meeting those requirements (Hauser and Clausing, 1988; Wasserman, 1993). It is also an important tool for concurrent engineering. In the era of globalization, the customer's order decoupling point (CODP) is at make-to-order (MTO) stage (Bhattacharya et al., 2005). From Figure 1 it is understood where to apply the QFD process. QFD is used at a CODP to ensure that the voice of the customer is heard throughout the product planning and design stage (Franceschini and Rosetto, 1995). QFD, in fact, is a method of continuous product improvement, emphasizing the impact of organizational learning on innovation (Govers, 2001).

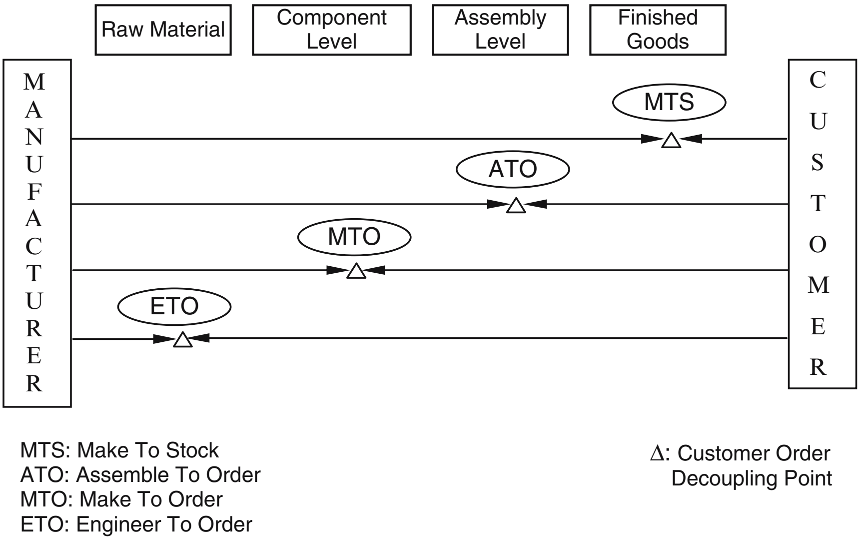

Figure 3. Relationship between CODP and MCDM-QFD process (Bhattacharya et al., 2005)

In QFD process, a matrix called the house-of-quality (HOQ) (Hauser and Clausing, 1988) is used to display the relationship between the voice of customers (WHATs) and the quality characteristics (HOWs) (Chuang, 2001). WHATs and HOWs are nothing but the customer and technical requirements, respectively. The HOQ is developed during the QFD transformation. Basically the HOQ demonstrates how the technical requirements satisfy the customer requirements. The matrix highlights the important issues in the planning of a new product or improving an existing product. QFD, when combining WHATs and HOWs with competitive analysis (WHYs), represents a customer-driven and market-oriented process for decision making (Cohen, 1995).

A traditional QFD model uses absolute importance to identify the degree of importance for each customer requirement. The psychology of customers, in general, is to rate almost everything as equally important, although it is not. As the absolute weighing data tend to be bunched near the highest possible scores, the differentiation of customer requirements is thus strongly recommended. These data, as they are, do not contribute much to helping QFD developers in prioritizing technical responses. At this juncture, the AHP (Saaty, 1988; 1990; 1994) prioritizes the customer's requirements by putting the relative degree of importance to each customer-requirement.

The task of the QFD team is to list the technical requirements (TRs). These requirements are most likely to affect the CRs. TR evaluators, in the QFD team, evaluate how the competitors' products compare with that of company's product. This evaluation leads to fixing of technical targets. From the QFD matrix, the discrepancies, if any, between the customers' perception and the QFD team's correlation of CR and TR can be easily understood. The vertical part of the QFD matrix shows how the company may respond to customer requirements.

4. DEVELOPMENT OF THE COMBINED AHP-QFD METHODOLOGY

The methodology integrating the MCDM methodology (AHP) and QFD for a selection problem comprises the following steps and is shown in Figure 4:

Step 1. Identification of customer requirements.

Step 2. Identification of technical requirements.

Step 3. Construction of central relationship matrix using expert knowledge of QFD team.

Step 4. Computation of degree of importance for customer requirements by using AHP.

Step 5. Computation of the degree of importance of technical requirements by Eq. (1).

$$w_j = \sum_{i=1}^{m} R_{ij} c_i \tag{1}$$

where

w_j = importance degree of the jth technical requirement $(\forall j = 1, 2, ..., n)$,

R_{ij} = quantified relationship between the ith customer requirement and the jth technical criteria in the central relationship matrix, and

c_i = importance weighing of the ith customer requirement.

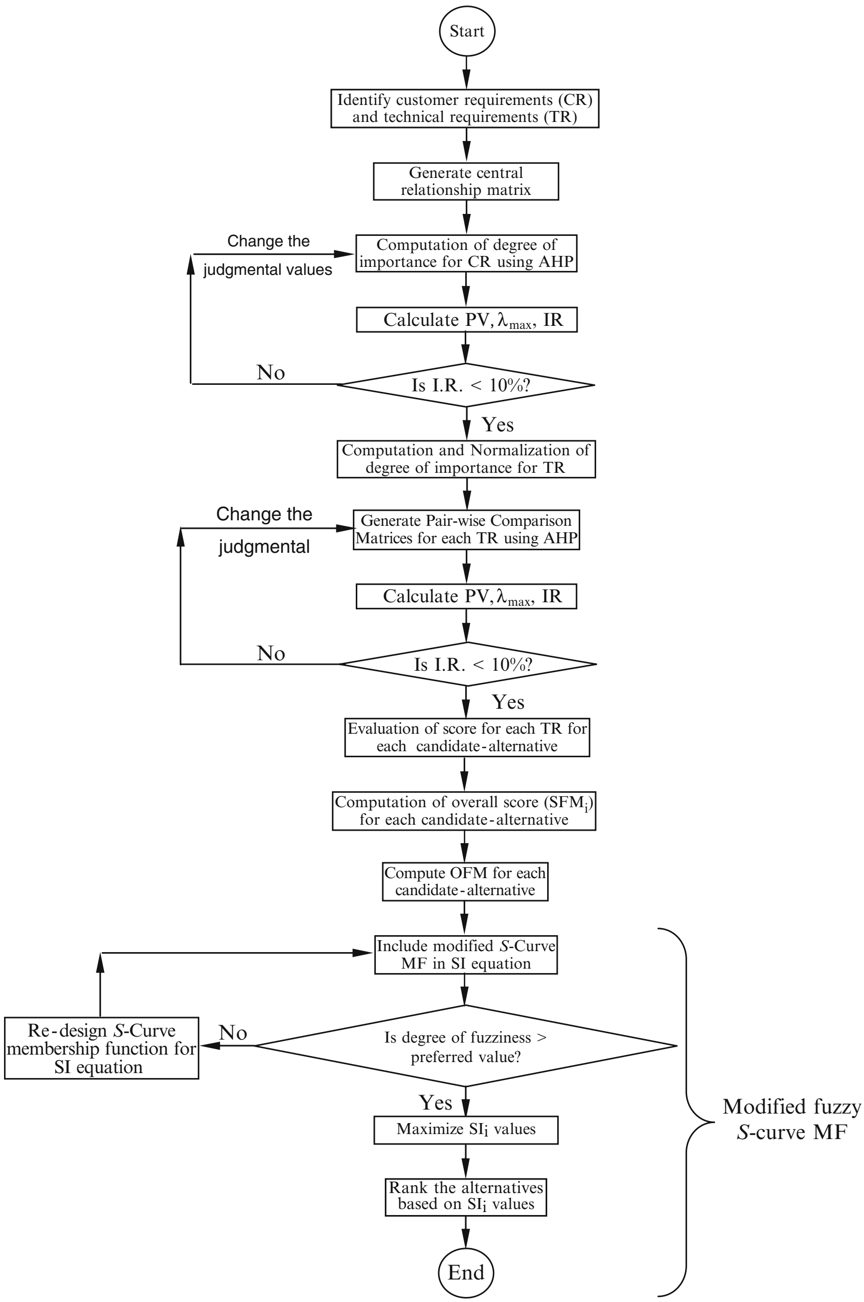

Figure 4. Flowchart of the proposed methodology

Step 6. Normalization of the degree of importance of technical criteria by Eq. (2).

$$\overline{w}_j = \frac{w_j}{\sum_{j=1}^{n} w_j} \times 100 \tag{2}$$

Step 7. Construction of pair-wise comparison matrices for each technical requirement using Saaty's (1988; 1990) nine-point scale.

Step 8. Evaluation of score, w_{ij} , for each technical requirement for each candidate-alternative.

Step 9. Computation of overall score (Chuang, 2001) by using Eq. (3).

$$S_j = \sum_{j=1}^{n} \overline{w}_j \, e_{ij} \tag{3}$$

where,

S_j = overall score for the jth candidate-alternative $(\forall j = 1, 2, ..., \mathrm{n})$,

$\overline{w}_j$ = normalized importance degree of the jth technical criteria $(j = 1, 2, ..., \mathrm{n})$, and

e_{ij} = PV value of the jth alternative on the ith technical criteria

Step 10. Computation of OFM values for each candidate robot by using Eq. 4.

OFM = Objective Factor Measure,

OFC = Objective Factor Cost,

SFM = Subjective Factor Measure,

SI = Selection Index,

α = Objective factor decision weight, and

n = number of candidate-alternatives (n = 4 in for the robot selection problem).

$$\mathrm{OFM}_i = [\ \mathrm{OFC}_i \times \sum(\ \mathrm{OFC}_i^{-1}\ )\ ]^{-1} \tag{4}$$

Step 11. Identification of fuzziness patterns and measurement of level-of- satisfaction of the decision maker using modified *S*-curve MF.

Step 12. Re designing the MF if the degree of fuzziness is greater than a preferred value.

Step 13. Maximization of the SI (selection index) value using Eq. 5.

$$\mathrm{SI}_i = [\ (\alpha \times \mathrm{SFM}_i\ ) + (\ 1 - \alpha\ ) \times \mathrm{OFM}_i\ ] \tag{5}$$

Step 14. Ranking of all the candidate-alternatives

Step 15. Selection of the best candidate-alternative using the analogy *the higher the score, the better the selection.*

5. ROBOT SELECTION PROBLEM

An illustrative example of a process industry dealing with an enormous volume of manufactured product was illustrated by Bhattacharya et al. (2005). Out of four robots, the best-suited robot was purchased for the desired job for a very specific manufacturing process using the methodology of combined AHP-QFD as depicted by Bhattacharya et al. (2005). But what is lacking in the said proposed model of Bhattacharya et al. (2005) is the evaluation of the fuzzy parameters in their multi-criteria selection model. When fuzzy parameters like human expertise and linguistic knowledge get involved with the model, there is always a need for the model to approximate the outputs within the required error rate. Thus, the ANFIS (Jang, 1991) is found suitable in dealing with this complex problem of multi-criteria decision making. Considering the robot selection data of Bhattacharya et al. (2005) we begin with fitting the modified S-curve MF (Eq. 6) in their methodology. Step 11 onward of the methodology have been proposed herein with the fuzzy S-curve MF.

$$\mu(x) = \begin{cases} 1 & x < x^a \\ 0.99 & x = x^a \\ \dfrac{B}{1 + Ce^{\gamma x}} & x^a < x < x^b \\ 0.001 & x = x^b \\ 0 & x > x^b \end{cases} \tag{6}$$

We use the previously identified customer requirements (CRs) viz., payload, accuracy, life-expectancy, velocity, programming flexibility and total cost of robot, and seven TRs, viz., drive system, geometrical dexterity, path measuring system, size, material, weight and initial operating cost of robot. As in the case of Bhattacharya et al. (2005) the job is to select the best one of the four robots. The additional purposes of the current model are to view the fuzziness patterns as well as the level-of-satisfaction of the decision maker, and to approximate the model with a predetermined allowable error rate.

For measuring the relative degree of importance for each customer requirement, based on the proposed methodology, a (6 × 6) decision matrix is constructed and shown in Figure 5.

$$\mathbf{D}=\begin{bmatrix}1 & 7 & 3 & 4 & 5 & 9\\ 1/7 & 1 & 1/3 & 1/2 & 2 & 3\\ 1/3 & 3 & 1 & 3 & 6 & 2\\ 1/4 & 2 & 1/3 & 1 & 3 & 4\\ 1/5 & 1/2 & 1/6 & 1/3 & 1 & 1/7\\ 1/9 & 1/3 & 1/2 & 1/4 & 7 & 1\end{bmatrix}=\begin{bmatrix}1 & 7 & 3 & 4 & 5 & 9\\ 0.143 & 1 & 0.333 & 0.500 & 2 & 3\\ 0.333 & 3 & 1 & 3 & 6 & 2\\ 0.250 & 2 & 0.333 & 1 & 3 & 4\\ 0.200 & 0.500 & 0.167 & 0.333 & 1 & 0.143\\ 0.111 & 0.333 & 0.500 & 0.250 & 7 & 1\end{bmatrix}$$

Figure 5. Decision matrix

The PV values of this decision matrix are found and $\lambda_{\max}$, I.I., R.I., and I.R. are calculated. If the level of inconsistency present in the information stored in the "D" matrix is satisfactory, the QFD team, then, puts the PV values in the transformation matrix. The next job of the QFD team is to find out the ranking of the given four robots based on the seven conflicting TRs. Seven pair-wise comparison matrices were built up based on the information on each TR.

Table 2. Overall Scores of the Four Robots

Technical Requirements	Weight	Importance weight for robots				I.I.	I.R.	Inconsistency (%)
		R_1	R_2	R_3	R_4			
1. Drive system	31.54	0.529	0.094	0.314	0.063	0.0249	0.0252	2.52
2.Geometrical dexterity	8.64	0.147	0.281	0.514	0.059	0.0116	0.0117	1.17
3. Path measuring system	9.47	0.074	0.520	0.105	0.300	0.0842	0.0851	8.51
4. Robot size	9.36	0.267	0.550	0.054	0.128	0.0644	0.0651	6.51
5. Material of robot	9.05	0.319	0.532	0.092	0.057	0.0866	0.0875	8.75
6. Weight of robot	26.46	0.523	0.089	0.326	0.062	0.0369	0.0373	3.73
7. Initial operating cost	5.48	0.483	0.086	0.355	0.077	0.0748	0.0756	7.56
Overall score		40.53	23.11	27.25	9.11			

Table 2 suggests R_1 » R_3 » R_2» R_4; i.e., R_1 gets precedence over R_3, which gets more importance over R_2 and R_4. Thus, the robot R_1 is selected as it has the highest overall score compared with others.

The total cost of the robotic system described in Bhattacharya et al. (2005) were broken down (refer to Table 3).

Table 3. Cost Factor Components and Their Units

Cost factor components	Range of attribute values
1. Acquisition cost of robot	US $ 4500 – 7000/unit
2. Cost of robot gripper mechanisms	US $ 2500 – 3000
3. Cost of sensors	US $ 900 – 1200
4. Total cost of layout necessary for installation of robot	US $ 3500 – 4000
5. Cost of feeders	US $ 400 – 900/unit
6. Maintenance cost	US $ 500 – 650/week
7. Cost of energy	US $ 6 – 10/Unit of electrical energy

The cost factors in Table 3 involve two types of costs, both a fixed and a recurring type. For four different robots, of which each can perform the very specified job, the attributes of the cost components are tabulated in Table 4.

Table 4. Attributes of Cost Factor Component

Robots → / Cost components ↓	R_1	R_2	R_3	R_4
1. Acquisition cost of robot	6500	5000	7000	4500
2. Cost of robot gripper mechanisms	2750	2500	3000	2900
3. Cost of sensors	1200	950	1100	1000
4. Total cost of layout	3650	4000	3875	3500
5. Cost of feeders	900	765	400	860
6. Maintenance cost	480	900	730	400
7. Cost of energy	7	8	10	6
Total (OFC) (US$)	15487	14123	16115	13166

A mathematical model was proposed by Bhattacharya et al. (2005) to combine cost factor components with the importance weightings found from AHP. The governing equation of the said model is

$$SI_i = [\, (\alpha \times SFM_i\,) + (\,1 - \alpha\,) \times OFM_i\,] \tag{7}$$

where,

$$\mathrm{OFM}_i = \frac{1}{\left[\mathrm{OFC}_i \times \sum_{i=1}^{n} \mathrm{OFC}^{-1}\right]}. \tag{8}$$

In the following chapters, we have discussed the implications of Eq. (8) as well as the modified S-curve MF with reference to the targeted MCDM modeling. Therefore, we refrain to discuss on these basic equations.

5.1 Computation of Level-of-Satisfaction, Degree of Fuzziness

We confine our efforts assuming that differences in judgmental values are only 5%. Therefore, the upper bound and lower bound of SFM_i as well as SI_i indices are to be computed within a range of 5% of the original value reported by Bhattacharya et al. (2005). In order to avoid complexity in delineating the technique proposed herein, we have considered, 5% measurement. One can fuzzify the SFM_i values from the very beginning of the AHP-QFD model by introducing a modified *S*-curve MF in AHP, and the corresponding fuzzification of SI_i indices can also be carried out using their holistic approach.

By using the equations above for a modified *S*-curve MF a relationship among the level-of-satisfaction of the decision maker, the degree of vaguenes and the SI indices is found. The results are plotted accordingly.

Figures 5a, b and c show three different plots depicting a relation among the level-of-satisfaction and SI indices for three different vagueness values. It should always be noted that higher the fuzziness, γ, values, the lesser will be the degree of vagueness inherent in the decision. Therefore, it is understood that the higher level of outcome of the decision variable, SI, for a particular level-of-satisfaction point, results in a lesser degree of fuzziness inherent in the said decision variable.

A relationship between the degree of fuzziness, γ, and the level-of-satisfaction α has been depicted by Figure 6. This is a clear indication that the decision variables, as defined in Eqs. (6) and (7), allows the MCDM model to achieve a higher level-of-satisfaction with a lesser degree of fuzziness.

Figures 7 and 8 delineate SI indices versus level-of-satisfaction α and SI indices versus degree of fuzziness γ, respectively. Now, let us examine the fuzziness inherent in each candidate-alternative.

There is a need to calculate both the upper bound and the lower bound solution of SI indices having a different level-of-satisfaction (α). The following figures have been found using MATLAB® version 7.0. The results have been encouraging, and the corresponding results have been indicated in Figures 9 to 14.

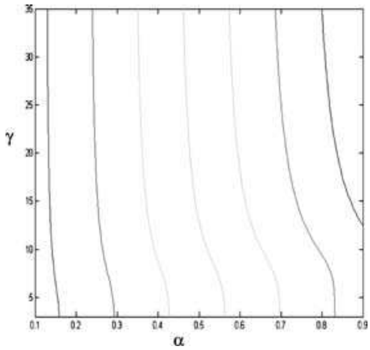

Figure 6. Fuzziness vs. α for Robot 1

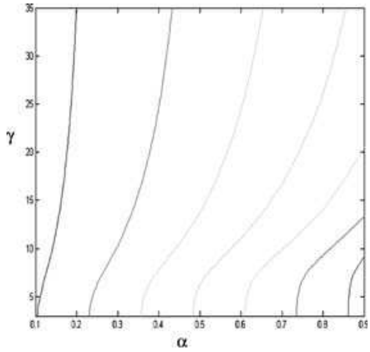

Figure 7. Fuzziness vs. α for Robot 2

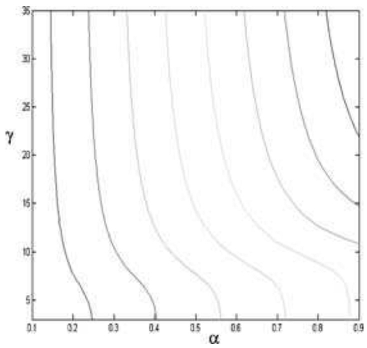

Figure 8. Fuzziness vs. α for Robot 3

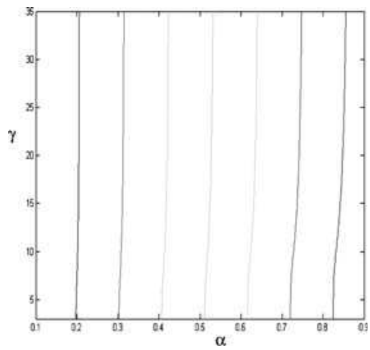

Figure 9. Fuzziness vs. α for Robot 4

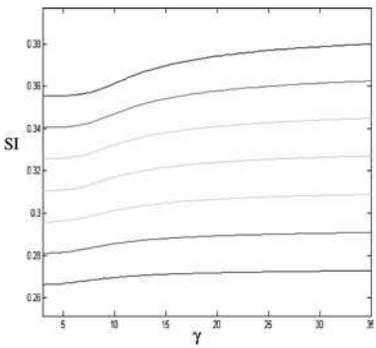

Figure 10. SI vs. γ for Robot 1

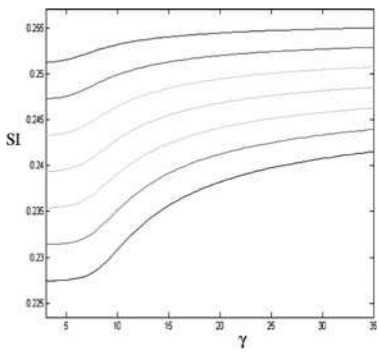

Figure 11. SI vs. γ for Robot 2

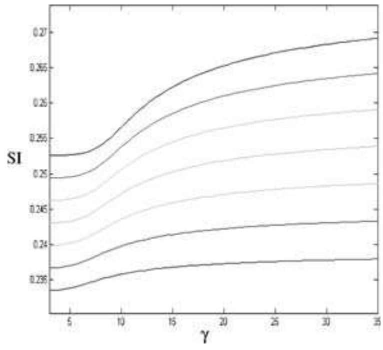

Figure 12. SI vs. γ for Robot 3

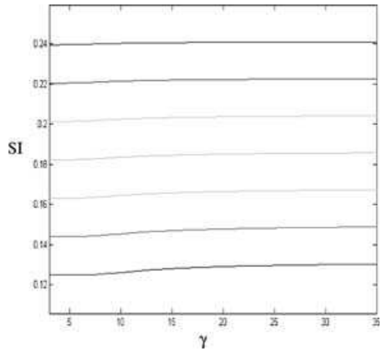

Figure 13. SI vs. γ for Robot 4

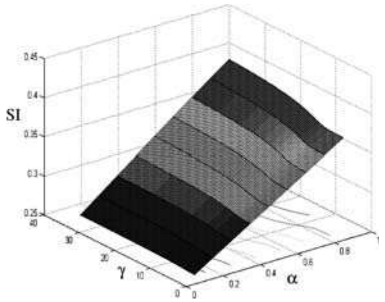

Figure 14. SI, γ, and α for Robot 1

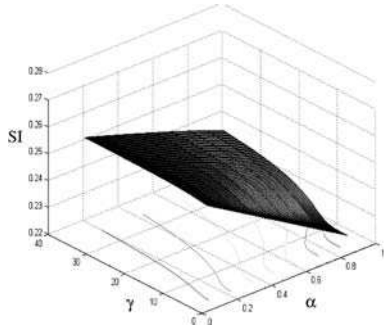

Figure 15. SI, γ, and α for Robot 2

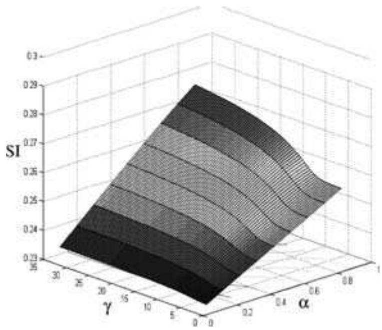

Figure 16. SI, γ, and α for Robot 3

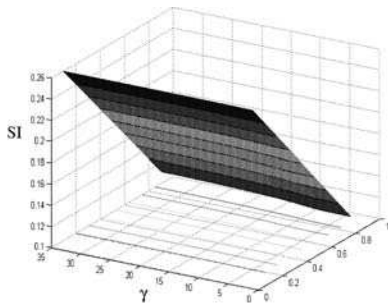

Figure 17. SI, γ, and α for Robot 4

Thus, the decision for selecting a candidate-alternative as seen from Figures 9 to 13 is tabulated in Table 5. It is noticed from the current investigation that this model eliciting the degree of fuzziness corroborates the MCDM model without fuzzification presented in Bhattacharya et al. (2005).

5.2 Experiment Results using the ANFIS Model

The experimental system consists of two stages: network training and performance evaluation. The task is to approximate the values of SI for different values of α and γ. In this chapter, we developed fuzzy inference systems for varying values of gamma keeping α = 0.001, 0.2, 0.4, 0.6, 0.8, and 1.0. Takagi Sugeno fuzzy inference was used with linear consequent parameters. We used four Gaussian MFs for the two variables α and γ. Sixteen fuzzy *if−then* rules were created during the neural learning process as depicted in Figures 18, 20, 22, 24, 26 and 28. The learned surfaces showing the input/output are illustrated in Figures 19, 21, 23, 25, 27 and 29. Empirical results are depicted in Table 5.

Table 5. Performance of the Fuzzy Inference Systems

α value	Root Mean Squared Error
0.001	0.0004
0.2	0.0009
0.4	0.0004
0.6	0.002
0.8	0.002
1.0	0.004

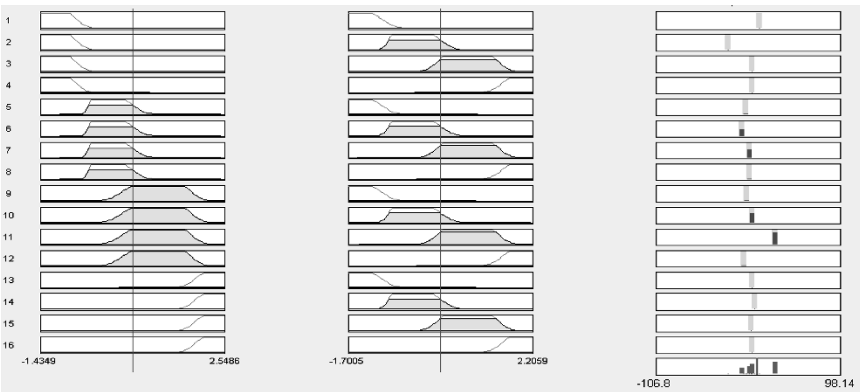

Figure 18. Developed Takagi Sugeno FIS (α = 0.001)

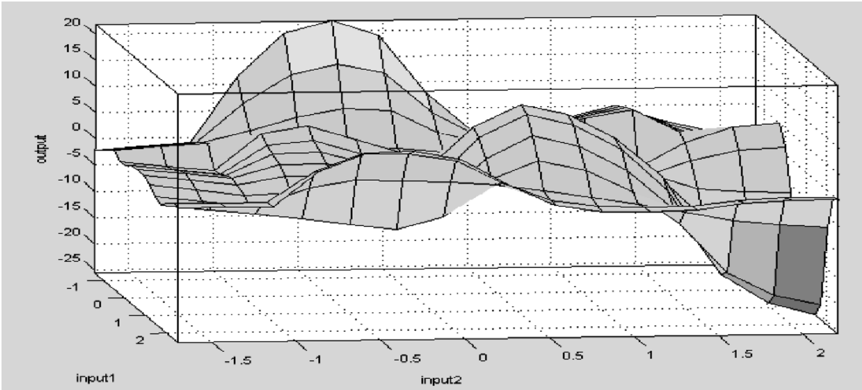

Figure 19. Input/Output surface mapping (α = 0.001)

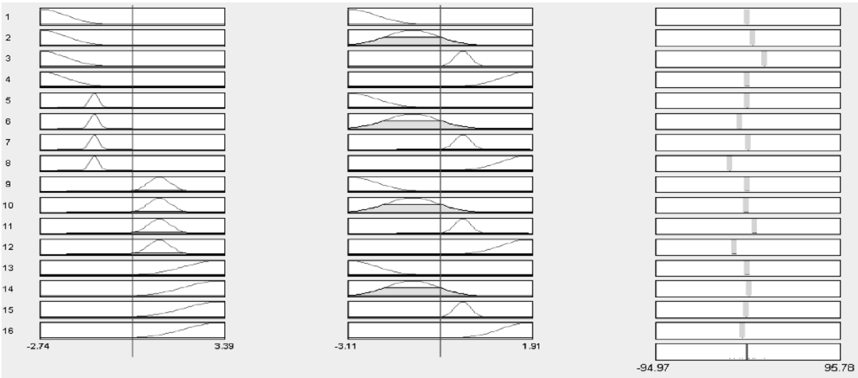

Figure 20. Developed Takagi Sugeno fuzzy inference system (α = 0.2)

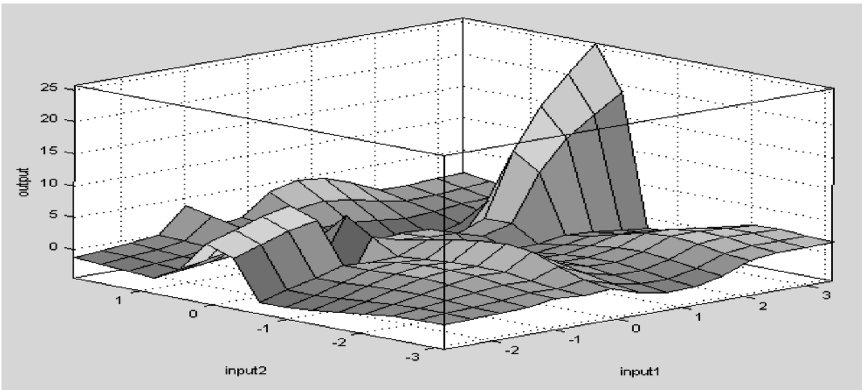

Figure 21. Input/Output surface mapping (α = 0.2)

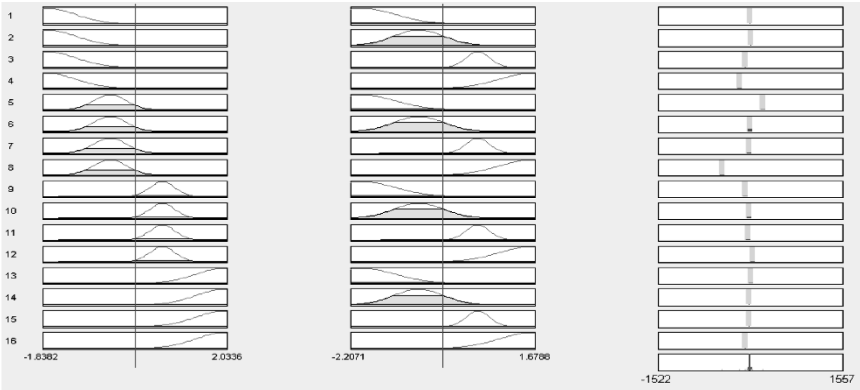

Figure 22. Developed Takagi Sugeno fuzzy inference system (α = 0.4)

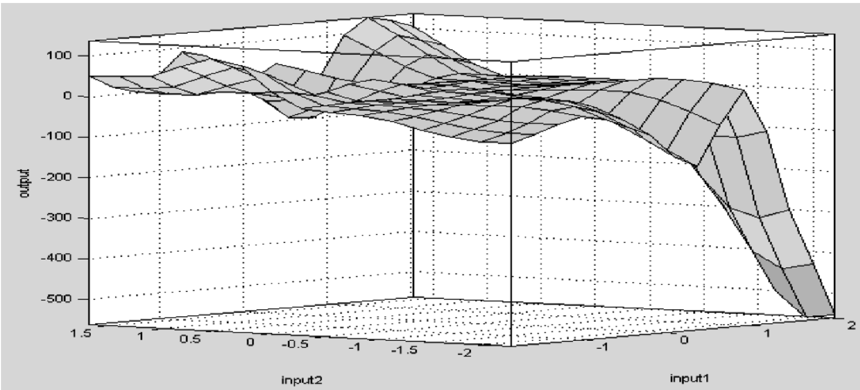

Figure 23. Input/Output surface mapping (α = 0.4)

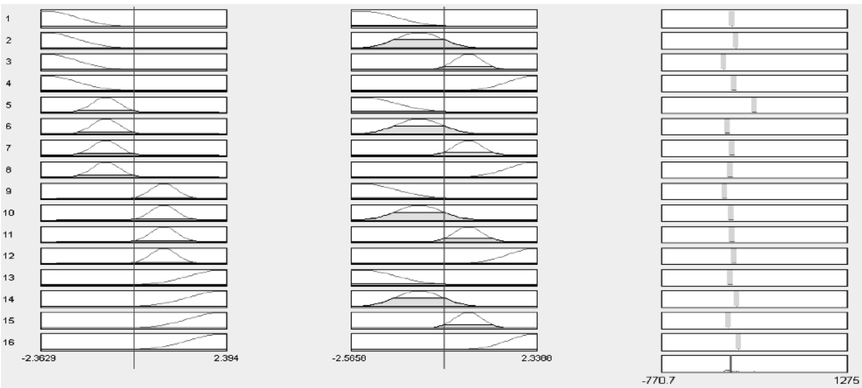

Figure 24. Developed Takagi Sugeno fuzzy inference system (α = 0.6)

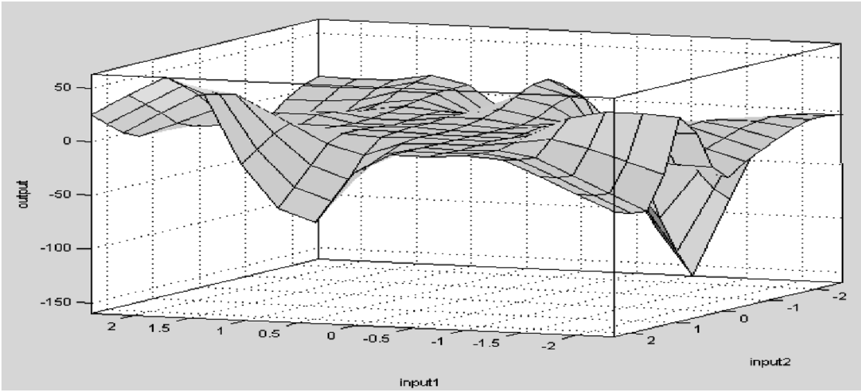

Figure 25. Input/Output surface mapping (α = 0.6)

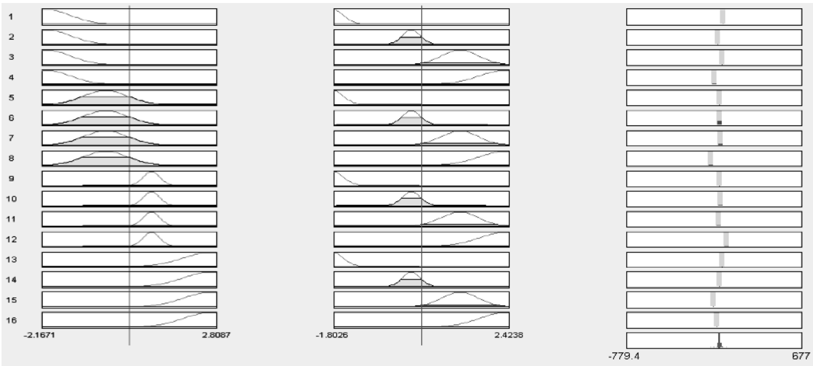

Figure 26. Developed Takagi Sugeno fuzzy inference system (α = 0.8)

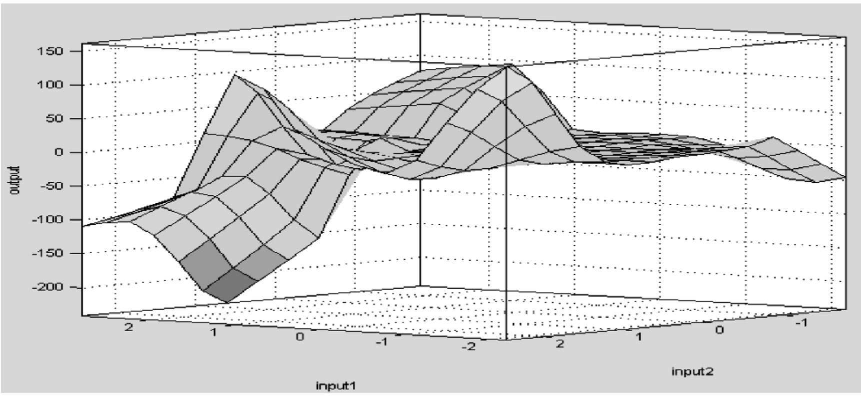

Figure 27. Input/Output surface mapping (α = 0.8)

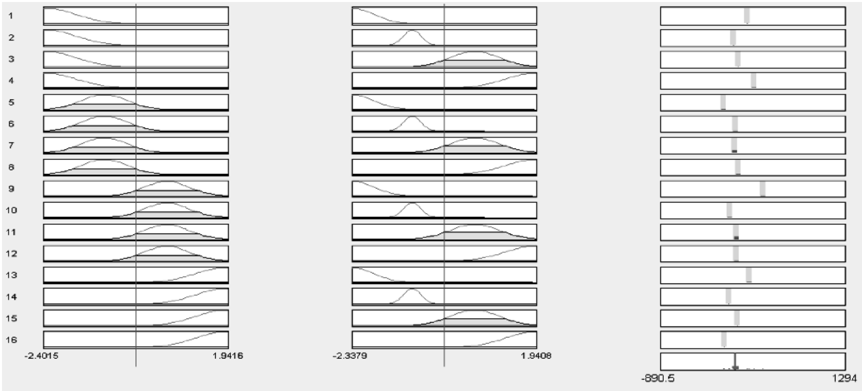

Figure 28. Developed Takagi Sugeno fuzzy inference system (α = 1.0)

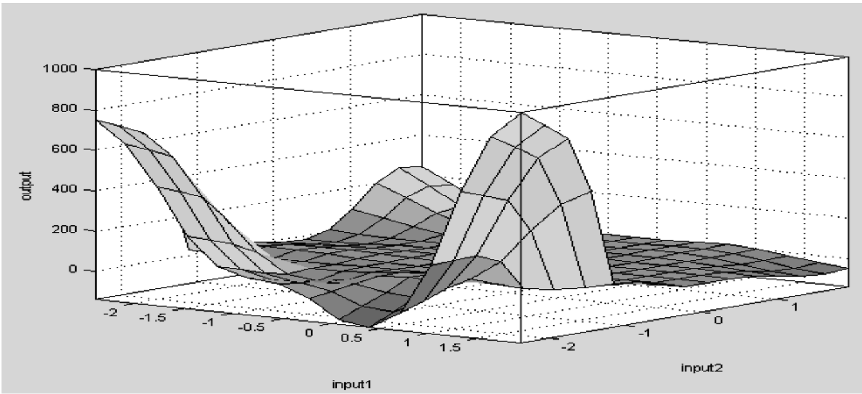

Figure 29. Input/Output surface mapping (α = 1.0)

6. DISCUSSION AND CONCLUSION

One underlying assumption of the proposed methodology is that the selection is made under certainty of the information data. In reality, the information available is highly uncertain and sometimes may be under risk also. The fuzzy *S*-curve MF helps in reducing the level of uncertainty as validated further by introducing the ANFIS model shown in Table 5. The root mean square errors as compared with the original values of level-of-satisfaction (α) are very low, and the satisfaction level of the decision makers are, thus, appreciable as well as within the acceptable level.

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FUZZY MULTIPLE OBJECTIVE LINEAR PROGRAMMING

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Abstract: In this chapter, first a literature review on the fuzzy multi-objective linear programming (FMOLP) and then its mathematical modeling with an application is given. FMOLP is one of the multi-objective modeling techniques most frequently used in the literature. The possible values of the parameters in FMOLP are imprecisely or ambiguously known to the experts. Therefore, it would be more appropriate for these parameters to be represented as fuzzy numerical data that can be represented by fuzzy numbers.

Key words: Multiple objectives, linear programming, interactive, approximation algorithm

1. INTRODUCTION

Multiple objective problems are concerned with the optimization of multiple, conflicting, and noncommensurable objective functions subject to constraints representing the availability of multiple objective problems that are concerned with the optimization of multiple, conflicting, and noncommensurable objective functions subject to constraints representing the availability of limited resources and requirements.

Multiple objective linear programming (MOLP) is one of the popular methods to deal with complex and ill-structured decision problems. When formulating an MOLP problem, various factors of the real world should be reflected in the description of the objective functions and the constraints. Naturally, these objective functions and constraints involve many parameters in which possible values may be assigned by the experts.

Normally, such parameters are set at some values in an experimental or subjective manner through the experts' understanding of the nature for the parameters (Sakawa, 1993).

The MOLP problem is specified by linear functions that are to be maximized subject to a set of linear constraints. The standard form of MOLP can be written as follows:

$$\text{Maximize } f(x) = Cx \tag{1}$$

$$\text{subject to } x \in X = \left\{ x \in R^n \,\middle|\, Ax \leq b,\, x \geq 0 \right\}$$

where C is an $k \times n$ objective function matrix, A is an $m \times n$ constraint matrix, $\tilde{b}$ is an m-vector of the right-hand side, and x is an n-vector of decision variables.

With this observation, it is natural to recognize that the possible values of these parameters are often imprecisely or ambiguously known to the experts. In this case, it may be more appropriate to interpret the experts' understanding of the parameters as fuzzy numerical data that can be represented by fuzzy numbers. The fuzzy multiple objective linear programming (FMOLP) problems involving fuzzy parameters would be viewed as a more realistic version than the conventional one (Sakawa, 1993). Various kinds of FMOLP models have been proposed to deal with different decision-making situations that involve fuzzy values in objective function parameters, constraints parameters, or goals.

Tanaka and Asai (1984) formulated FMOLP with triangular fuzzy numbers, and the nonlinear programming problem obtained was solved by using a max–min operator. Luhandjula (1987) proposed the concept of α-possible feasibility and β-possible efficiency and resolved imprecise objectives and constraints with fuzzy numbers by solving an auxiliary crisp MODM problem derived by using the extension principle and α-and β-level cuts. Korhonen et al. (1989) propose a general approach to semi-structured decision making, which makes it possible to consider multiple objectives (flexible gloals), "hard" constraints (inflexible: goals), and "soft" constraints (fuzzy goals) within the same framework. Rommelfanger et al. (1989) present a new method calleld "alpha-level related pair formation" for solving linear programming problems with fuzzy parameters in the objective function. Lai and Hwang (1992) resolved imprecise objectives with triangle fuzzy numbers with maximizing the most possible value, minimizing the risk of obtaining lower profit, and maximizing the possibilities of obtaining higher profit, and they used a fuzzy ranking concept to resolve imprecise constraints. Slowinski and Tenghem (1993) compare the methods fuzzy linear programming (FLIP) and strategy for nuclear generation of electricity (STRANGE) developed by themselves, respectively. Zimmermann (1993) surveys major models and theories in mathematical programming and offers some indication on the expected future developments. Turtle et al. (1994) show how fuzzy logic can be employed using straightforward LP tools. Julien (1994) investigates the application of fuzzy set and possibility theories for the representation of imprecise information in water quality management problems. Fuller and Fedrizzi (1994) explore stability analysis in possibilistic programming by extending previous research results to possibilistic linear programs with multiple objective functions. They use multi-objective possibilistic linear programs with continuous fuzzy number coefficients. Nakahara and Gen (1994) propose a quantitative formulation of LP problems with fuzzy number coefficients, by using the ranking criteria proposed by themselves, and show an algorithm for solving the formulated problems in some cases. The optimization of an objective function with fuzzy number coefficients is formulated as the user-oriented extension of the optimization of an objective function with real coefficients by the proposed ranking criteria. Carlsson and Fuller (1995) introduce measures of interdependence between the objectives in order to provide for a better understanding of the decision problems and to find effective and more correct solutions into multiple criteria decision-making problems. Sakawa et al. (1995) show that large-scale fuzzy LP problems can be reduced to a number of independent linear sub-problems (and the overall satisfying solution for the decision maker is directly obtained just solving the sub-problems. Herrera and Verdegay (1995) study some models for dealing with fuzzy integer LP problems that have a certain lack of precision of a vague nature in their formulation and present methods to solve them with either fuzzy constraints or fuzzy numbers in the objective function or fuzzy numbers defining the set of constraints.

Kahraman et al. (1996) propose a fuzzy multi-objective linear programming that considers intangible benefits in AMTs and expands the constraints by adding tolerances. The transition from vagueness to quantification is performed by applying the fuzzy set theory. The approach also considers the vagueness in the objective functions by using the membership functions. The main advantage of the fuzzy LP, compared with the unfuzzy problem formulation, is the fact that the decision maker is not forced into a precise formulation for mathematical reasons.

Downing and Ringuest (1998) implement four multi-objective linear programming algorithms on microcomputer software packages and conduct a large field experiment using the implemented algorithms. Two new algorithms that incorporate formal models of decision maker behavior are tested along with two established algorithms that include no formal models of decision-maker behavior.

Borges and Antunes (2002) study the effects of uncertainty on multiple objective linear programming models using the concepts of fuzzy set theory. The proposed interactive decision support system is based on the interactive exploration of the weight space. The comparative analysis of indifference regions on the various weight spaces (which vary according to intervals of values of the satisfaction degree of objective functions and constraints) enables study of the stability and evolution of the basis that correspond to the calculated efficient solutions with changes of some model parameters.

Wang and Liang (2004) develop an FMOLP model for solving the multi-product aggregate production planning (APP) decision problem in a fuzzy environment. The proposed model attempts to minimize total production costs, carrying and backordering costs, and rates of changes in labor levels considering inventory level, labor levels, capacity, warehouse space, and the time value of money.

Jana and Chattopadhyay (2004) design a model of energy utilization by developing a decision support frame for an optimized solution to the problem, taking into consideration four sources and six devices suitable for the study area, namely Narayangarh Block of Midnapore District in India. Since the data available from rural and unorganized sectors are often ill-defined and subjective in nature, many coefficients are fuzzy numbers, and hence several constraints appear to be fuzzy expressions. In this study, the energy allocation model is initiated with three separate objectives for optimization, namely minimizing the total cost, minimizing the use of non-local sources of energy, and maximizing the overall efficiency of the system. Since each of the above objective-based solutions has relevance to the needs of the society and economy, it is necessary to build a model that makes a compromise among the three individual solutions.

El-Ela et al. (2005) present a proposed procedure that depends on the multi-objective fuzzy linear programming (MFLP) technique to obtain the optimal preventive control actions, for power generation and transmission line flows, to overcome any emergency conditions. The proposed multi-objective functions are minimizing the generation cost function, maximizing the generation reserve at certain generator, maximizing the generation reserve for all generation system, and maximizing the preventive action for one or more critical transmission line.

Jana and Roy (2005) present the solution procedure for a multi-objective fuzzy linear programming problem (MOFLPP) with mixed constraints and its application in solid transportation problem. There are two parts in this paper. In the first part, a multi-objective linear programming problem with fuzzy coefficients occurring in constraints and objective functions and fuzzy constraint goals is considered. Fuzzy constraint goals and coefficients of objective and constraint functions are characterized by triangular fuzzy numbers (TFNs). Using Bellman and Zadeh's (1970) multi-criteria fuzzy decision-making process, the very problem is converted to a crisp non linear programming problem. Then it is solved using a fuzzy decisive set method. In the other part, a linear multi-objective solid transportation problem with mixed constraint as well as an additional restriction in a fuzzy environment is considered. In this transportation problem, the cost coefficients of objective functions and the additional restriction function as well as the supply, demand, and conveyance capacity are expressed as TFNs. This MOFLPP is solved by the fuzzy decisive set method as in the first part.

Wu et al. (2006) develop a new approximate algorithm for solving FMOLP problems involving fuzzy parameters in any form of membership functions in both objective functions and constraints. Liang (2006) develops an interactive fuzzy multi-objective linear programming (i-FMOLP) method for solving the fuzzy multi-objective transportation problems with a piece-wise linear membership function.

The interactive FMOLP method includes the following steps (Liang, 2006):

Step 1. Formulate the original fuzzy MOLP model for the considered problem.

Step 2. Given the minimum acceptable membership level, α, and then convert the fuzzy inequality constraints with fuzzy available resources (the right-hand side) into crisp ones using the weighted average method.

Step 3. Specify the degree of membership for several values of each objective function.

Step 4. Draw the piece-wise linear membership functions for each objective function.

Step 5. Formulate the piece-wise linear equations for each membership function.

Step 6. Introduce an auxiliary variable, thus enabling the original fuzzy multi-objective problem to be aggregated into an equivalent ordinary LP form using the minimum operator.

Step 7. Solve the ordinary LP problem, and execute the interactive decision process. If the decision maker is dissatisfied with the initial solutions, the model must be adjusted until a set of satisfactory solutions is derived.

Li et al. (2006) improve the fuzzy compromise approach of Guu and Wu (1999) by automatically computing proper membership thresholds instead of choosing them. In practice, choosing membership thresholds arbitrarily may result in an infeasible optimization problem. Although they can adjust minimum satisfaction degree to get a fuzzy efficient solution, it sometimes makes the process of interaction more complicated. In order to overcome this drawback, a theoretically and practically more efficient two-phase max–min fuzzy compromise approach is proposed.

2. FUZZY MULTI-OBJECTIVE LINEAR PROGRAMMING

When all coefficients of the objective functions and the constraints are fuzzy number parameters represented in any form of membership functions, such FMOLP problems can be formulated as follows:

$$\text{Maximize } \tilde{f}(x) = \tilde{C}x \tag{2}$$

$$\text{subject to } x \in X = \left\{ x \in R^n \,\middle|\, \tilde{A}x \leq \tilde{b},\, x \geq 0 \right\}$$

where $\tilde{C}$ is an $k \times n$ matrix, each element of which $\tilde{c}_{ij}$ is a fuzzy number $\mu_{\tilde{c}_{ij}}(x)$, represented by membership function $\mu_{\tilde{c}_{ij}}(x)$; $\tilde{A}$ is an $m \times n$ matrix, each element of which $\tilde{a}_{ij}$ is a fuzzy number represented by membership function $\mu_{\tilde{a}_{ij}}(x)$; $\tilde{b}$ is an m-vector, each element of which $\tilde{b}_i$ is a fuzzy number represented by membership function $\mu_{\tilde{b}_i}(x)$; and x is an n-vector of decision variables, $x \in R^n$.

Associated with the FMOLP problems (Eq. 2), the following MOLP_λ problems can be written as (Wu et al., 2006):

$$\text{Maximize } \begin{pmatrix} C_\lambda^L x \\ C_\lambda^R x \end{pmatrix}, \ \forall \lambda \in [0,1]$$

$$\text{subject to} \tag{3}$$

$$x \in X = \left\{ x \in R^n \,\middle|\, A_\lambda^L x \le b_\lambda^L, A_\lambda^R x \le b_\lambda^R, x \ge 0, \forall \lambda \in [0,1] \right\}$$

where

$$C_\lambda^L = \begin{bmatrix} c_{11\lambda}^L & c_{12\lambda}^L & \cdots & c_{1n\lambda}^L \\ c_{21\lambda}^L & c_{22\lambda}^L & \cdots & c_{2n\lambda}^L \\ \vdots & \vdots & \ddots & \vdots \\ c_{k1\lambda}^L & c_{k2\lambda}^L & \cdots & c_{kn\lambda}^L \end{bmatrix}$$

$$C_\lambda^R = \begin{bmatrix} c_{11\lambda}^R & c_{12\lambda}^R & \cdots & c_{1n\lambda}^R \\ c_{21\lambda}^R & c_{22\lambda}^R & \cdots & c_{2n\lambda}^R \\ \vdots & \vdots & \ddots & \vdots \\ c_{k1\lambda}^R & c_{k2\lambda}^R & \cdots & c_{kn\lambda}^R \end{bmatrix}$$

$$A_\lambda^L = \begin{bmatrix} a_{11\lambda}^L & a_{12\lambda}^L & \cdots & a_{1n\lambda}^L \\ a_{21\lambda}^L & a_{22\lambda}^L & \cdots & a_{2n\lambda}^L \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1\lambda}^L & a_{m2\lambda}^L & \cdots & a_{mn\lambda}^L \end{bmatrix}$$

$$A_\lambda^R = \begin{bmatrix} a_{11\lambda}^R & a_{12\lambda}^R & \cdots & a_{1n\lambda}^R \\ a_{21\lambda}^R & a_{22\lambda}^R & \cdots & a_{2n\lambda}^R \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1\lambda}^R & a_{m2\lambda}^R & \cdots & a_{mn\lambda}^R \end{bmatrix}$$

$$b_\lambda^L = \left[b_{1\lambda}^L, \quad b_{2\lambda}^L, \quad \cdots, \quad b_{m\lambda}^L \right]^T$$

$$b_\lambda^R = \left[b_{1\lambda}^R, \ b_{2\lambda}^R, \ \cdots, \ b_{m\lambda}^R \right]^T$$

Wu et al. (2006) propose an approximation algorithm as follows for solving MOLP_λ problem, the solution of which is equally the solution of FMOLP problem.

$$\text{Maximize} \quad \gamma \tag{4}$$

subject to

$$\frac{c_{i\lambda_j}^L x - f_{i\lambda_j}^{L\min}}{f_{i\lambda_j}^{L\max} - f_{i\lambda_j}^{L\min}} \geq \gamma\,, \qquad \frac{c_{i\lambda_j}^R x - f_{i\lambda_j}^{R\min}}{f_{i\lambda_j}^{R\max} - f_{i\lambda_j}^{R\min}} \geq \gamma$$

$$a_{s\lambda_j}^L x \leq b_{s\lambda_j}^L\,, \qquad a_{s\lambda_j}^R x \leq b_{s\lambda_j}^R$$

where

$$f_{i\lambda_j}^{L_{\max}} = c_{i\lambda_j}^L x_{i\lambda_j}^{L*}, \ i = 1, 2, ..., k; \ j = 0, 1, ..., l,$$
$$f_{i\lambda_j}^{R_{\max}} = c_{i\lambda_j}^R x_{i\lambda_j}^{R*}, \ i = 1, 2, ..., k; \ j = 0, 1, ..., l,$$

$$f_{i\lambda_j}^{L\min} = \min_{\substack{s=1,\dots,k \\ t=1,\dots,l \\ s\neq i, t\neq j}} \left(c_{i\lambda_j}^L x_{s\lambda_t}^{L*}, c_{i\lambda_j}^L x_{s\lambda_t}^{R*} \right)$$

$$f_{i\lambda_j}^{R\min} = \min_{\substack{s=1,\dots,k \\ t=0,1,\dots,l \\ s\neq i, t\neq j}} \left(c_{i\lambda_j}^R x_{s\lambda_t}^{L*}, c_{i\lambda_j}^R x_{s\lambda_t}^{R*} \right)$$

and x^* is said to be an optimal solution.

2.1 A Numerical Example

Assume that you have two fuzzy linear objective functions and four fuzzy linear constraints as follows.

$$\text{Max} \quad \tilde{f}(x) = \max \begin{pmatrix} \tilde{f}_1(x) = \tilde{c}_{11}x_1 + \tilde{c}_{12}x_2 \\ \tilde{f}_2(x) = \tilde{c}_{21}x_1 + \tilde{c}_{22}x_2 \end{pmatrix}$$

subject to

$$\tilde{a}_{11}\, x_1 + \tilde{a}_{12}\, x_2 \ \tilde{\leq}\ \tilde{b}_1, \quad \tilde{a}_{21}\, x_1 + \tilde{a}_{22}\, x_2 \ \tilde{\leq}\ \tilde{b}_2$$

$$\mu_{\tilde{c}_{11}} = \begin{cases} 0, \ x \leq 5 \\ \left(x^2 - 25\right)/24, \ 5 \leq x \leq 7 \\ 1, \ 7 \leq x \leq 9 \\ \left(100 - x^2\right)/19, \ 9 \leq x \leq 10 \\ 0, \ 10 \leq x \end{cases}$$

$$\mu_{\tilde{c}_{12}} = \begin{cases} 0, \ x \leq 2 \\ \left(x - 2\right)/5, \ 2 \leq x \leq 7 \\ 1, \ 7 \leq x \leq 12 \\ \left(196 - x^2\right)/52, \ 12 \leq x \leq 14 \\ 0, \ 14 \leq x \end{cases}$$

$$\mu_{\tilde{c}_{21}} = \begin{cases} 0, \ x \leq 14 \\ \left(x^2 - 196\right)/188, \ 14 \leq x \leq 18 \\ 1, \ 18 \leq x \leq 22 \\ \left(576 - x^2\right)/92, \ 22 \leq x \leq 24 \\ 0, \ 24 \leq x \end{cases}$$

$$\mu_{\tilde{c}_{22}} = \begin{cases} 0, & x \le 30 \\ \left(x^2 - 900\right)\big/325, & 30 \le x \le 35 \\ 1, & 35 \le x \le 40 \\ \left(2025 - x^2\right)\big/425 & 40 \le x \le 45 \\ 0, & 45 \le x \end{cases}$$

$$\mu_{\tilde{a}_{11}} = \begin{cases} 0, & x \le 0 \\ 2x, & 0 \le x \le 0.5 \\ 1, & 0.5 \le x \le 2 \\ \left(25 - 5x\right)/15, & 2 \le x \le 5 \\ 0, & 5 \le x \end{cases}$$

$$\mu_{\tilde{a}_{12}} = \begin{cases} 0, & x \le 0 \\ 5x, & 0 \le x \le 0.2 \\ 1, & 0.2 \le x \le 1 \\ \left(9 - 3x\right)/6, & 1 \le x \le 3 \\ 0, & 3 \le x \end{cases}$$

$$\mu_{\tilde{a}_{21}} = \begin{cases} 0, & x \le 0 \\ 4x, & 0 \le x \le 0.25 \\ 1, & 0.25 \le x \le 5 \\ \left(64 - 8x\right)/24, & 5 \le x \le 8 \\ 0, & 8 \le x \end{cases}$$

$$\mu_{\tilde{a}_{22}} = \begin{cases} 0, \ x \le 5 \\ (2x-10)/2, \ 5 \le x \le 6 \\ 1, \ 6 \le x \le 7 \\ (64-8x)/8, \ 7 \le x \le 8 \\ 0, \ 8 \le x \end{cases}$$

$$\tilde{b}_1 = \begin{cases} 0, \ x \le 12 \\ (x-12)/3, \ 12 \le x \le 15 \\ 1, \ 15 \le x \le 18 \\ (42-2x)/6, \ 18 \le x \le 21 \\ 0, \ 21 \le x \end{cases}$$

$$\tilde{b}_2 = \begin{cases} 0, \ x \le 56 \\ (2x-112)/24, \ 56 \le x \le 68 \\ 1, \ 68 \le x \le 74 \\ (258-3x)/36, \ 74 \le x \le 86 \\ 0, \ 86 \le x \end{cases}$$

This fuzzy MOLP problem can be solved by the normal Simplex algorithm.

3. CONCLUSIONS

Multiple objective problems are concerned with the optimization of multiple, conflicting, and noncommensurable objective functions subject to constraints representing the availability of limited resources and requirements. In this chapter an example of FMOLP is provided in which all coefficients are the objective functions and the constraints are fuzzy number parameters represented in any form of membership functions. The fuzzy multi-objective LP provides great flexibility to make the estimates of the problem parameters. The main advantage of the fuzzy LP, compared with the unfuzzy problem formulation, is the fact that the decision maker is not forced into a precise formulation because of mathematical reasons.

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QUASI-CONCAVE AND NONCONCAVE FMODM PROBLEMS

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Abstract: A membership function may be concave-shaped or convex-shaped. In this chapter, first, concave and convex membership values are analyzed and, in practice, commonly used approaches for solving an fuzzy multi-objective decision-making (FMODM) problem are briefly reviewed. Then, some proposition and remarks are presented to solve a quasi-concave FMODM problem. The proposed method can directly solve a quasi-concave FMODM problem by using standard LP techniques.

Key words: Quasi-concave, nonconcave, LP

1. INTRODUCTION

Decision making (DM) is part of people lives, and the history of DM even dates back to before the dawn of history, but the scientific research of a systematic procedure of describing the human DM process appeared in the early 1960s (Simon, 1960). Although the DM theory proposed by Simon has received a lot of attention, the DM process described by Simon lies on single objective only. In many real decision situations, more than one objective has to be considered and different kinds of uncertainty must be handled (Abdelaziz et al., 2004), particularly in multi-dimension, multi-criterion, and nondominated perspective DM problems. Therefore, multi-objective decision-making (MODM) programming have long drawn a wide spectrum of attention from both academicians and practitioners. Typically, MODM starts with determining a set of objectives and ends with finding the best acceptable

C. Kahraman (ed.), Fuzzy Multi-Criteria Decision Making.

solution based on the decision-maker preferences. Examples of MODM include purchasing a car, recruiting a new manager, choosing the best portfolio, appointing a spokesperson, or underwriting an insurance contract.

Because of human beings' inherent subjectivity, imprecision, and vagueness in expressing judgments, in practice, decision makers may frequently express their evaluations in a form of uncertainty rather than preciseness. Since fuzzy theory is very helpful in dealing with fuzziness of human judgment quantitatively, using fuzzy theory to treat MODM has been discussed since the 1970s (Yu, 2001). However, fuzzy multi-objective decision making (FMODM) problems have been noticed worldwide since the work published by Zimmermann (1981) where Zimmermann introduced conventional linear programming and multi-objective linear programming into fuzzy set theory. Since then, various methods using linear programming (LP) have been developed to solve FMODM problems.

An FMODM problem is usually formulated to maximize and/or minimize several objectives simultaneously subject to a constraint set. Accordingly, a general FMODM problem, in which the aggregated goal is the minimum operator of individual goals, is formulated as follows:

FMODM Problem:

Maximize λ (1.1)
subject to $\lambda \leq \mu_i(z_i)$, $i = 1, 2, ..., n$
$\mu_i(z_i) = |z_i(X) - g_i|$, $z_i(X) \in F$ (a feasible set),

where $\mu_i(z_i)$ is the membership function of the ith objective function, g_i denotes the fuzzy goal of the ith objective function, $z_i(X)$ is the ith objective function, and X is a vector of decision variables.

Many studies (Biswal, 1997; Hannan, 1981a, 1981b; Mjelde, 1983; Nakamura, 1984; Narasimhan, 1980; Romero, 1994; Yang et al., 1991) indicate that most real-world applications in engineering, physical, business, social, and management fields are not pure linear, triangular, concave, or convex FMODM problems but rather quasi-concave or more general nonconcave FMODM problems. Due to one of the most promising techniques of linearizing non concave functions is piece-wise linear programming. Hence, FMODM problems with piece-wise linear membership functions have been studied by Narasimhan (1980), Hannan (1981), Nakamura (1984), Inuiguchi et al. (1990), and Yang et al. (1991).

A membership function $\mu_i(z_i)$ may be concave-shaped or convex-shaped, as shown in Figure 1(a) and (b), respectively. The marginal possibility with respect to a concave membership function is decreasing, whereas the marginal possibility with respect to a convex membership function is increasing. If the marginal possibility increases first, then decreases, or decreases first then increases, then the membership function becomes a convex–concave or concave−convex mixed shape as shown in Figure 1(c). In practice, membership functions are not concave or convex but the mixed shapes composed by concave and convex curves or even more general non concave curves as shown in Figure 1(d).

2. REVIEW OF FMODM PROGRAMMING

Commonly used approaches for solving a FMODM problem in Eq. 1 are briefly reviewed in this section. In 1980, Narasimham proposed a LP approach to solving a FMODM problem with triangular membership functions, as expressed below.

FMODM Model 1 (Narasimham Method):

Maximize λ

subject to $\lambda \leq \mu_i(z_i)$, $i = 1, 2, \ldots, n$

$$\mu_i(z_i) = \begin{cases} 0 & \text{if } g_i \geq b_i + d_i \\ 1 - \dfrac{g_i - b_i}{d_i} & \text{if } b_i \leq g_i \leq b_i + d_i \\ 1 - \dfrac{b_i - g_i}{d_i} & \text{if } b_i - d_i \leq g_i \leq b_i \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

$z_i(X) \in F$ (a feasible set),

where $\mu_i(z_i)$ is the membership function of the ith objective function, g_i denotes the ith fuzzy goal, $z_i(X)$ is the ith objective function, X is a vector of decision variables, d_i is a chosen positive constant for the maximum allowable deviation from the aspiration level of the ith goal, and b_i is the most desired value of the ith objective function.

Two primary drawbacks in Narasimham's method are listed below:

1. An FMODM problem has to be divided into 2^n sub-problems where n is the number of goals.
2. All membership functions are restricted in triangular or trapezoidal shapes.

Extending triangular or trapezoidal to general concave-shaped membership functions, Hannan (1981) presented a method for converting a FMODM problem in Eq. 1 into the following model:

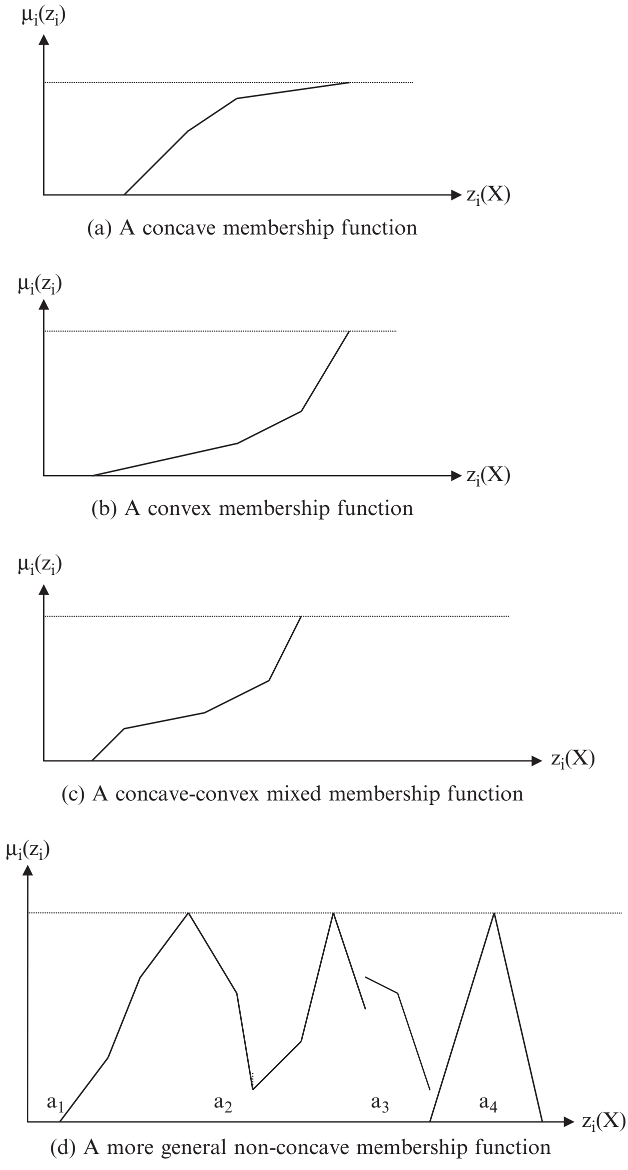


Figure 1. Membership Functions

FMODM Model 2 (Hannan Method):

Maximize λ (2)

subject to $\lambda \le \mu_i(z_i)$, $i = 1, 2, ..., n$ $\quad z_i + d_{ij}^{-} - d_{ij}^{+} = g_{ij}$,

$z_i \in F,\ d_{ij}^- \geq 0,\ d_{ij}^+ \geq 0,\ j = 1, 2, ..., N_i$

where $\mu_i(z_i)$, a concave typed function, is expressed as

$\mu_i(z_i) = \sum_{j=1}^{N_i} \alpha_{ij} \mid z_i - g_{ij} \mid + \beta_i z_i + r_i \quad \mid z_i - g_{ij} \mid = d_{ij}^- + d_{ij}^+$ in which α_{ij}, β_i, and r_i are parameters, g_{ij} are the change points of segments, d_{ij}^- and d_{ij}^+ are deviation variables.

The serious limitation in Hannan's method is that all $\mu_i(z_i)$ should be concave functions. For tackling a quasi-concave FMOP problem, Inuiguchi et al. (1990) developed a approach of transforming all quasi-concave functions into concave functions. Consider the following example slightly modified from Inuiguchi et al. (1990).

Example 1:

Maximize λ

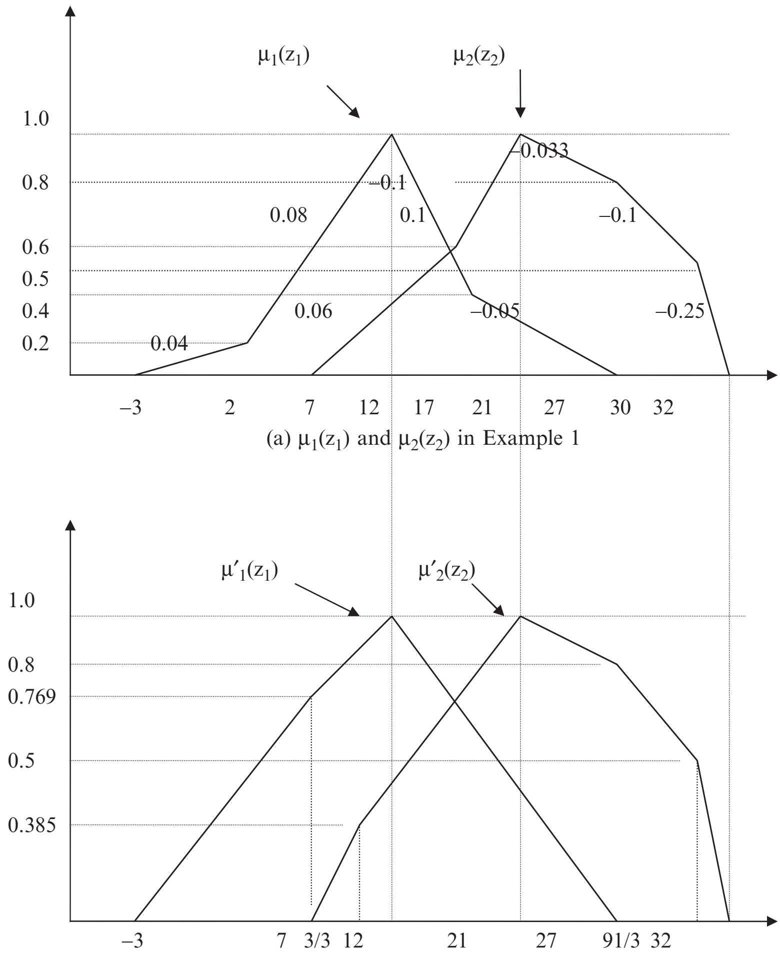
subject to $\lambda \leq \mu_1(z_1)$, $\lambda \leq \mu_2(z_2)$,

$z_1 = -x_1 + 2x_2$, $z_2 = 2x_1 + x_2$,
$-x_1 + 3x_2 \leq 21$, $x_1 + 3x_2 \leq 27$,
$4x_1 + 3x_2 \leq 45$, $3x_1 + x_2 \leq 30$,
$x_1,\ x_2 \geq 0$

$$\mu_1(z_1) = \begin{cases} 0, & z_1 \leq -3 \\ 0.04z_1, & -3 \leq z_1 \leq 2 \\ 0.08z_1 + 0.2, & 2 \leq z_1 \leq 12 \\ 1, & z_1 = 12 \\ -0.1z_1 + 2.2, & 12 \leq z_1 \leq 17 \\ -0.05z_1 + 0.5, & 17 \leq z_1 \leq 27 \\ 0, & z_1 \geq 27 \end{cases}$$

$$\mu_2(z_2) = \begin{cases} 0, & z_2 \leq 7 \\ 0.06z_2, & 7 \leq z_2 \leq 17 \\ 0.1z_2 + 0.6, & 17 \leq z_2 \leq 21 \\ 1, & z_2 = 21 \\ -0.033z_2 + 1.7, & 21 \leq z_2 \leq 27 \\ -0.1z_2 + 0.8, & 27 \leq z_2 \leq 30 \\ -0.25z_2 + 0.5, & 30 \leq z_2 \leq 32 \\ 0, & z_2 \geq 32 \end{cases}$$

where $\mu_1(z_1)$ and $\mu_2(z_2)$ are specified in Figure 2(a).



(b) Two converted $\mu'_1(z_1)$ and $\mu'_2(z_2)$ in Inuiguchi et al. Model

Figure 2. Membership values

Notably both $\mu_1(z_1)$ and $\mu_2(z_2)$ are quasi-concave functions, as depicted in Figure 2(a). Inuiguchi et al. first convert $\mu_1(z_1)$ and $\mu_2(z_2)$ into two concave functions $\mu'_1(z_1)$ and $\mu'_2(z_2)$, respectively, as shown in Figure 2(b). Example 1 then can be solved by the following LP model:

FMODM Model 3 (Inuiguchi et al. Method for Example 1):

$$
\begin{aligned}
&\text{Maximize} \quad \lambda' \\
&\text{subject to } \lambda' \le \mu_1'(z_1), \lambda' \le \mu_2'(z_2) \\
&z_1 = -x_1 + 2x_2,\ z_2 = 2x_1 + x_2 \\
&\qquad -x_1 + 3x_2 \le 21, x_1 + 3x_2 \le 27 \\
&4x_1 + 3x_2 \le 45,\ 3x_1 + x_2 \le 30, x_1,\ x_2 \ge 0
\end{aligned}
$$

$$
\mu_1'(z_1) = \begin{cases}
0, & z_1 \le -3 \\
\min\left(\frac{1}{13}z_1 + \frac{3}{13},\ \frac{3}{65}z_1 + \frac{29}{65}\right), & -3 \le z_1 \le 12 \\
1, & z_1 = 12 \\
-\frac{1}{15}z_1 + \frac{9}{5}, & 12 \le z_1 \le 27 \\
0, & z_1 \ge 27
\end{cases}
$$

$$
\mu_2'(z_2) = \begin{cases}
0, & z_2 \le 7 \\
\min\left(\frac{3}{26}z_2 - \frac{21}{26},\ \frac{3}{52}z_2 - \frac{11}{52}\right), & 7 \le z_2 \le 17 \\
1, & z_2 = 21 \\
\min\left(-\frac{1}{5}z_2 + \frac{32}{3}, -\frac{1}{15}z_2 + \frac{8}{3}, -\frac{1}{45}z_2 + \frac{53}{45}\right), & 21 \le z_2 \le 32 \\
0, & z_2 \ge 32.
\end{cases}
$$

Although Inuiguchi et al.'s idea is very useful in formulating quasi-concave functions into concave functions, there are three shortcomings in Inuiguchi et al. method as described below:

If the number of break points is large, then it causes a tedious computational burden to convert these membership functions into concave functions.

That transforming procedure is complicated and cannot effectively deal with an FMODM problem with more general nonconcave functions.

That method still requires zero-one variables to treat converted concave functions [i.e., $\mu_1'(z_1)$ and $\mu_2'(z_2)$].

Take Example 1, for instance: Five break points are required to do transforming computing. Suppose there are *n* objective functions and each of these functions having *mi* break points, then the number of transforming computing is

$$\sum_{i=1}^{n} m_i$$

The situation would become more complicated for treating more general nonconcave FMODM problems. Consequently, Yang et al. (1991)

presented another method for treating a quasi-concave FMODM problem. Take Example 1, for instance. Yang et al.'s method could formulate Example 1 as following a zero-one programming model [as depicted in Figure 3(a) and 3(b)]:

FMOP Model 4 (Yang et al.'s method for Example 1)

Maximize λ

subject to $\lambda \le 1 - \frac{a_4 - z_1}{d_1} + M(1-\delta_1) + M\delta_2$

$$\lambda \le 1 - \frac{12 - z_1}{d_2} + M\delta_1 + M\delta_2, \qquad \lambda \le 1 - \frac{a_3 - z_1}{d_3} + M\delta_1 + M\delta_2$$

$$\lambda \le 1 - \frac{27 - z_1}{d_4} + M(1-\delta_2) + M\delta_1, \qquad \lambda \le 1 - \frac{a_6 - z_2}{d_5} + M(1-\delta_3)$$

$$\lambda \le 1 - \frac{21 - z_2}{d_6} + M\delta_3, \lambda \le 1 - \frac{a_{10} - z_2}{d_7} + M\delta_3, \qquad \lambda \le 1 - \frac{a_9 - z_2}{d_8} + M\delta_3$$

$$\lambda \le 1 - \frac{32 - z_2}{d_9} + M\delta_3, \qquad z_1 = -x_1 + 2x_2$$

$$z_2 = 2x_1 + x_2, \qquad -x_1 + 3x_2 \le 21$$

$$x_1 + 3x_2 \le 27, \qquad 4x_1 + 3x_2 \le 45$$

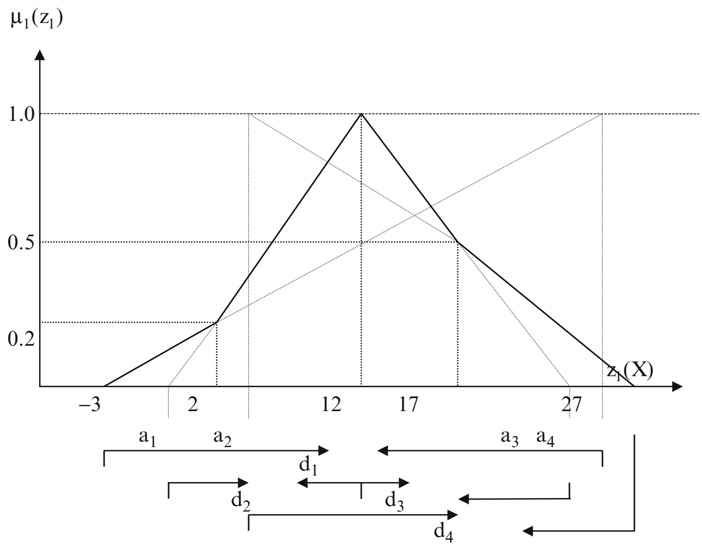
$$3x_1 + x_2 \le 30, \qquad x_1, \ \mathrm{x}_2 \ge 0$$

where δ_1, δ_2, and δ_3 are zero−one variables; M is a big number; and a_1, a_2, …, a_{10} are approximated end-point values as depicted in Figure 3(a) and 3(b).

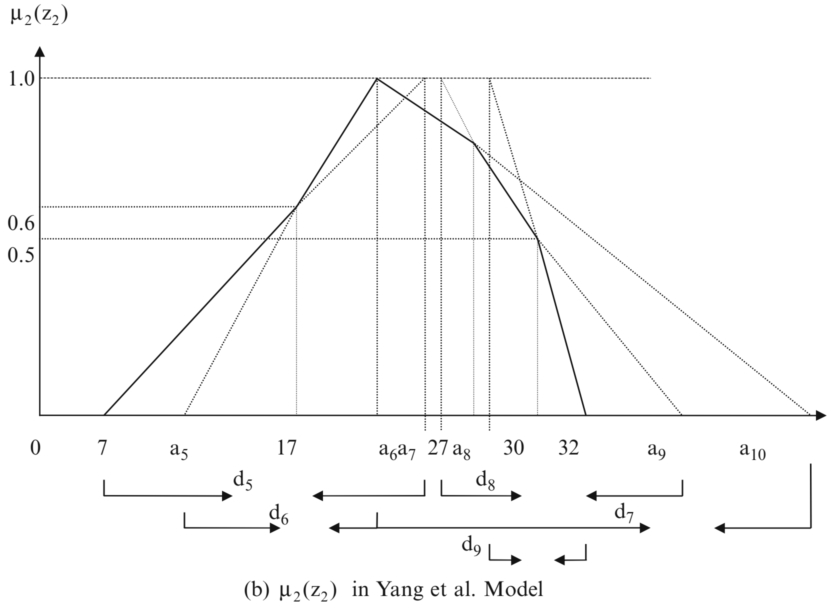
A major disadvantage in Yang et al.'s method (1991) is that it involves too many zero-one variables for treating quasi-concave FMODM problems. The number of zero-one variables equals the number of intersections between convex and concave functions. In addition, many end-point approximations are required before formulating a quasi-concave FMODM program.

Take Example 1, for instance, $\mu_i(z_i)$ contains two convex−concave intersections and $\mu_2(z_2)$ contains one convex−concave intersection. Therefore, three zero−one variables (i.e., δ_1, δ_2, or δ_3) are added in the solution process. In addition, ten times end-point approximations (i.e., a_1, a_2, …, a_{10}) are required in formulating FMODM model 2. A detailed discussion is given in Li and Yu (1999).

Considering $\mu_i(z_i)$ in Problem (1.1) could be concave, convex, or concave-convex mixed−type functions, Nakamura developed a method to



(a) $\mu_1(z_1)$ in Yang et al. Model



(b) $\mu_2(z_2)$ in Yang et al. Model

Figure 3. Membership Values

expres a general piece−wise membership function (1984). He reformulates Problem (1.1) as follow:

FMODM Model 5 (Nakamura Method):

Maximize λ

subject to $\lambda \le \mu_{\tilde{D}}(z_i)$ for $i = 1, 2, ..., n$

where $\mu_{\tilde{D}}(z_i) = [\{(\overset{im'}{\underset{j=1}{\vee}} \sigma_j(z_i)) \wedge \overset{im''}{\underset{j=1}{\vee}} \rho_j(z_i)\} \wedge \{\overset{im'''}{\underset{j=1}{\wedge}} \sigma_j(z_i)\} \wedge 1] \vee 0$ in which $j = 1, 2, ..., i_{m'}$ are the change points over the convex part of each $\mu_i(z_i)$, $j = 1, 2, ..., i_{m''}$ are the number of linear functions for separating concave or convex parts over each $\mu_i(z_i)$, $j = 1, 2, ..., i_{m'''}$ are the change points over concave part of each $\mu_i(z_i)$, and $\sigma_i(z_i)$ are linear functions representing part of $\mu_i(z_i)$.

Nakamura's method encounters two major difficulties:

Expression of piece-wise membership functions is intricate; it requires repetitive use of an LP computation for solving an FMOP problem.

That method divides an FMOP problem into $\prod_{i=1}^{n} 2k_i$ sub-problems and requires $2\sum_{i=1}^{n} k_i$ constraints, where k_i is the number of segments for each $\mu_i(z_i)$.

Take Example 1, for instance, Nakamura expresses the membership functions, depicted in Figure 4(a) and 4(b), as follows:

$$\mu_1(z_1) = [\{\sigma_1(z_1) \vee \sigma_2(z_1)\} \wedge \{\rho_1(z_1)\} \wedge \{\sigma_3(z_1) \vee \sigma_4(z_1)\} \wedge 1] \vee 0$$

$$\mu_2(z_2) = [\{\sigma_5(z_2) \vee \sigma_6(z_2)\} \wedge \rho_2(z_2) \wedge \sigma_7(z_2) \wedge \sigma_8(z_2) \wedge \sigma_9(z_2) \wedge 1] \vee 0$$

where $\vee$ stands for maximum, $\wedge$ stands for minimum, $\{\sigma_1(z_1) \vee \sigma_2(z_1)\}$, $\{\sigma_3(z_1) \vee \sigma_4(z_1)\}$, and $\{\sigma_5(z_2) \vee \sigma_6(z_2)\}$ are the sets of the convex parts.

Nakamura's method then divides Example 1 into eight sub-problems. Some of these sub-problems are expressed as follows:

FMODM Model 6 (Nakamura Method for Example 1):

Sub-problem 1:

Maximize λ

subject to $\lambda \le \sigma_1(z_i) \wedge \rho_1(z_1) \wedge \sigma_3(z_1)$

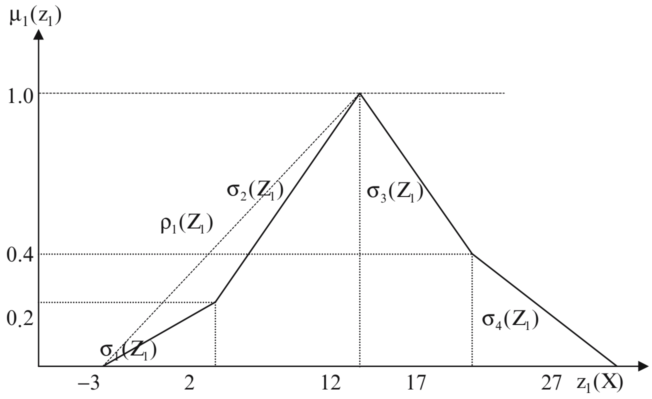
$\lambda \le \sigma_5(z_2) \wedge \rho_2(z_2) \wedge \sigma_7(z_2) \wedge \sigma_8(z_2) \wedge \sigma_9(z_2)$

Sub-problem 2:

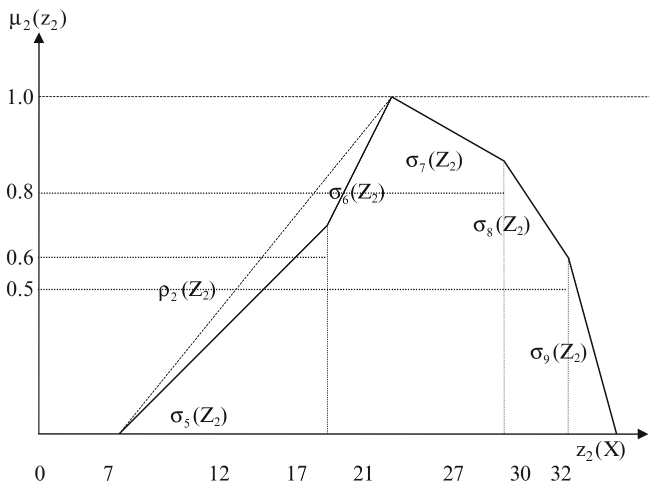
Maximize λ

subject to $\lambda \le \sigma_2(z_i) \wedge \rho_1(z_1) \wedge \sigma_3(z_1)$

$$\lambda \le \sigma_5(z_2) \wedge \rho_2(z_2) \wedge \sigma_7(z_2) \wedge \sigma_8(z_2) \wedge \sigma_9(z_2)$$



(a) $\mu_1(z_1)$ in Nakamura's Model



(b) $\mu_2(z_2)$ in Nakamura's Model

Figure 4. Membership functions

Sub-problem 6:

Maximize λ

subject to $\lambda \le \sigma_2(z_i) \wedge \rho_1(z_1) \wedge \sigma_3(z_1)$

$\lambda \le \sigma_6(z_2) \wedge \rho_2(z_2) \wedge \sigma_7(z_2) \wedge \sigma_8(z_2) \wedge \sigma_9(z_2)$:

After using LP computation repeatedly, Nakamura's method finds the optimal solution in Sub-problem 6. To solving Example 1, Nakamura's method involves eight sub-problems and finds the optimal solution in Subproblem 6 after using the LP computation repeatedly. As a result, Nakamura's method encounters two major difficulties:

Expression of piece-wise membership functions is intricate; it requires repetitive use of the LP computation for solving an FMOP problem.

Nakamura's method has to divide an FMOP problem into $\prod_{i=1}^{n} 2k_i$ sub-problems and requires $2\sum_{i=1}^{n} k_i$ constraints, where k_i is the number of segments for each $\mu_i(z_i)$.

3. PROPOSED METHOD

Building on the above discussion, this section first presents a convenient way to interpret a piece-wise linear membership function. The proposed expression is simpler than Nakamura's method (1984). An FMODM problem in Equation (1) with piece-wise quasi-concave functions is termed a quasi-concave FMODM problem. Some propositions and remarks, presented by Yu and Li (2000), for solving a quasi-concave FMODM problem are described as follows.

PROPOSITION 1.

Let $\mu_i(z_i)$ be a piece-wise linear membership function of $z_i(X)$, as depicted in Figure 5(a), where a_k, k = 1, 2, ..., m are the break points of $\mu_i(z_i)$, s_k, k = 1, 2, ..., m-1 are the slopes of line segments between a_k and a_{k+1}, and

$$s_k = \frac{\mu_i(a_{k+1}) - \mu_i(a_k)}{a_{k+1} - a_k}$$

$\mu_i(z_i)$ can then be expressed as:

$$\mu_i(z_i) = \mu_i(a_1) + s_1(z_i(X) - a_1) + \sum_{k=2}^{m-1} \frac{s_k - s_{k-1}}{2}(|z_i(X) - a_k| + z_i(X) - a_k) \quad (3)$$

where |o| is the absolute value of 0.

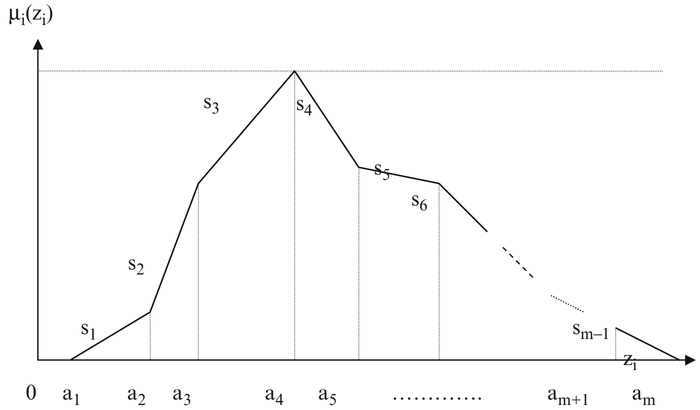


Figure 5. Membership functions

This proposition can be examined as follows: (Proof)

(i) If $z_i(X) \leq a_2$, then

$$\mu_i(z_i) = \mu_i(a_1) + \frac{\mu_i(a_2) - \mu_i(a_1)}{a_2 - a_1}(z_i(X) - a_1) = a_1 + s_1(z_i(X) - a_1)$$

(ii) If $z_i(X) \leq a_3$, then

$$\mu_i(z_i) = \mu_i(a_1) + s_1(a_2 - a_1) + s_2(z_i(X) - a_2)$$

$$= \mu_i(a_1) + s_1(z_i(X) - a_1) + \frac{s_2 - s_1}{2}(|z_i(X) - a_2| + z_i(X) - a_2)$$

(iii) If $z_i(X) \leq a_{k'}$, then $\sum_{k \geq k'}^{m-1}(|z_i(X) - a_k| + z_i(X) - a_k) = 0$ and $\mu_i(z_i)$ becomes

$$\mu_i(a_1) + s_1(z_i(X) - a_1) + \sum_{k=2}^{k'-1} \frac{s_k - s_{k-1}}{2}(|z_i(X) - a_k| + z_i(X) - a_k)$$

Take $\mu_1(z_1)$ and $\mu_2(z_2)$ in Example 1 [as depicted in Figure 2(a)] for instance, $\mu_1(z_1)$ and $\mu_2(z_2)$ can be represented by Proposition 1 as

$$\mu_1(z_1) = \begin{pmatrix} 0.04(z_1+3) + \frac{0.08-0.04}{2}|z_1-2| + (z_1-2) \\ + \frac{-0.1-0.08}{2}(|z_1-12|+z_1-12) + \frac{-0.05+0.1}{2}(|z_1-17|+z_1-17) \end{pmatrix} \tag{4}$$

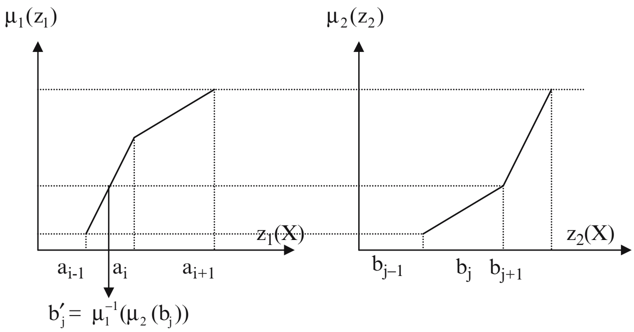
$$\mu_2(z_2) = \begin{pmatrix} 0.06(z_2-7) + \frac{0.1-0.06}{2}(|z_2-17|+z_2-17) + \\ \frac{-0.033-0.1}{2}\begin{pmatrix}|z_2-21| \\ +z_2-21\end{pmatrix} + \frac{-0.1+0.033}{2}(|z_2-27|+z_2-27) \\ + \frac{-0.25+0.1}{2}(|z_2-30|+z_2-30) \end{pmatrix} \tag{5}$$

An advantage of expressing a quasi-concave membership function by Eq. (3) is the convenience of knowing the intervals of convexity and concavity for $\mu_i(z_i)$, as described below:

Remark 1 (Define a convex-type break point). For a $\mu_i(z_i)$ expressed by Eq. (3), if $s_{k+1} > s_k$, then $\mu_i(z_i)$ is a convex function for $a_{k-1} \le z_i(X) \le a_{k+1}$ and a_k is called a convex-type break point of z_i. Take Eq. (4) for instance, it is convenient to check that $\mu_1(z_1)$ is concave when $2 \le z_1(X) \le 17$ and that $\mu_1(z_1)$ is convex when $-3 \le z_1(X) \le 12$ and $12 \le z_1(X) \le 27$. Therefore, the point $z_1(X)=2$ and $z_1(X)=17$ are convex-type break points of z_i. Similarly for Equation (6), $\mu_2(z_2)$ is convex for $7 \le z_2(X) \le 21$ and concave for $17 \le z_2(X) \le 32$. $z_2(X)=17$ is a convex-type break point of z_2.

Remark 2 (Define a concave-type break point). For a $\mu_i(z_i)$ expressed by Eq. (3) if $s_{k+1} < s_k$ then $\mu_i(z_i)$ is a concave function for $a_{k-1} \le z_i(X) \le a_{k+1}$ and a_k is called a concave-type break point of z_I.

Remark 3 (Define a mapping point). For $\mu_1(z_1)$ and $\mu_2(z_2)$ shown in Figure 6(b) and 6(c), respectively, we can find a convex-type break point b_j in z_2 by using Remark 1. Then a corresponding point of b_j can be found in z_1 which has the same value of membership functions as b_j. Such a point is called a *mapping point* of b_j, denoted as b'_j, which is mapped from z_2 to z_1 and is calculated by $b'_j = \mu_1^{-1}(\mu_2(b_j))$.



(a) a concave function $\mu_1(z_1)$ (b) a convex function $\mu_2(z_2)$

Figure 6. Concave and convex functions

Remark 4 (Specify a converted concave function). Now let us consider two piece-wise linear functions μ_1 and μ_2 specified in Figure 6(a).

$$\mu_1(f(X)) = \mu_1(a_1) + s_1(f(X) - a_1) + \frac{s_2 - s_1}{2}(|f(X) - a_2| + f(X) - a_2) \tag{6}$$

$$\mu_2(f(X)) = \mu_2(b_1) + t_1(f(X) - b_1) + \frac{t_2 - t_1}{2}(|f(X) - b_2| + f(X) - b_2) \tag{7}$$

where $s_1 > s_2 > 0$, $t_2 > t_1 > 0$, $a_1 = b_1$, and $a_3 = b_3$. Then two converted concave functions μ'_1 and μ'_2, shown in Figure 6(b), can be specified as follows:

$$\mu'_1(f(X)) = \begin{pmatrix} \mu'_1(a_1) + s_3(f(X) - a_1) + \dfrac{s_4 - s_3}{2}(|f(X) - b'_2| + \\ f(X) - b'_2) + \dfrac{s_5 - s_4}{2}(|f(X) - a_2| + f(X) - a_2) \end{pmatrix} \tag{8}$$

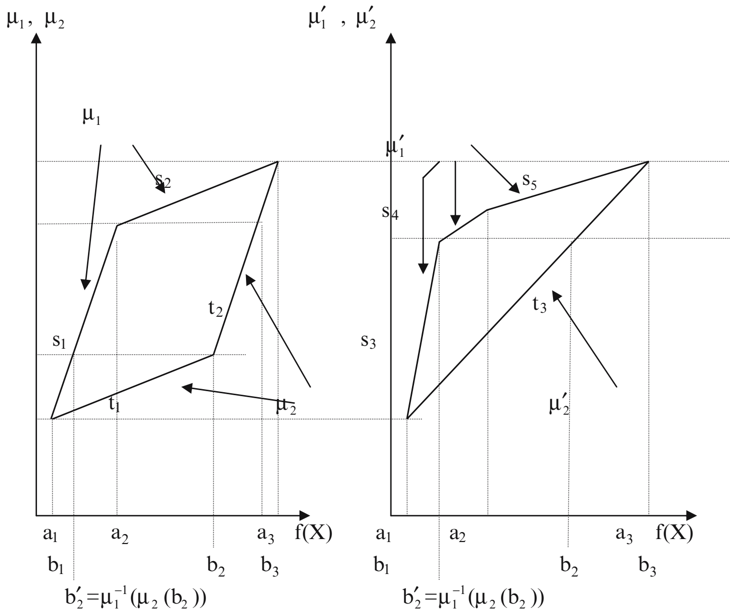
$$\mu'_2(f(X)) = \mu'_2(b_1) + t_3(f(X) - b_1) \tag{9}$$

$$\text{where } a_1 = b_1,\ a_3 = b_3,\ t_3 > 0,\ s_3 > s_4 > s_5 > 0, \tag{10}$$

$$\left.\begin{aligned} \mu_1(a_1) = \mu'_1(a_1) = 0,\ \mu_1(a_3) = \mu'_1(a_3) = 1 \\ \mu_2(b_1) = \mu'_2(b_1) = 0,\ \mu_2(b_3) = \mu'_2(b_3) = 1 \end{aligned}\right\} \tag{11}$$

$$\frac{\mu'_1(b'_2)}{\mu'_2(b_2)} = \frac{\mu_1(b'_2)}{\mu_2(b_2)} = 1 \tag{12}$$

$$\text{and } b_2' = \mu_1^{-1}[\mu_2(b_2)]. \tag{13}$$



(a) Two piecewise linear membership functions μ_1 and μ_2

(b) Two piecewise linear membership functions μ_1' and μ_2'

Figure 7. Membership functions

Next, Proposition 2 is presented below:

PROPOSITION 2.

Function μ_1' specified in Eqs. (9)−(13) is a concave function.

Proof: Due to $s_3 > s_4 > s_5$, based on Remark 2, b_2' and a_2 become concave-type points on μ_1'. Consequently, μ_1' is a concave function.

Consider the following example:

Example 2:

Maximize λ

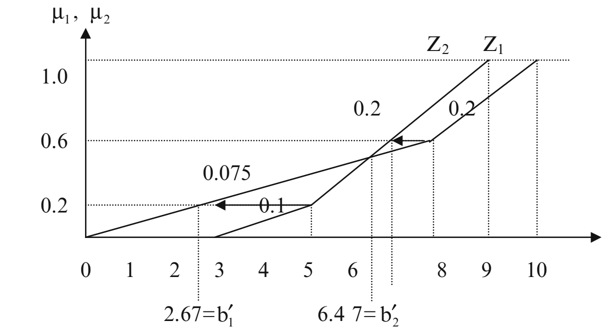
Subject to: $\lambda \le \mu_1(z_1)$, $\lambda \le \mu_2(z_2)$,

$z_1 = -x_1 + 2x_2$, $z_2 = 2x_1 + x_2$, $-x_1 + 3x_2 \le 6$, $x_1 + 3x_2 \le 12$, $4x_1 + 3x_2 \le 30$, $3x_1 + x_2 \le 15$, $x_1, x_2 \ge 0$

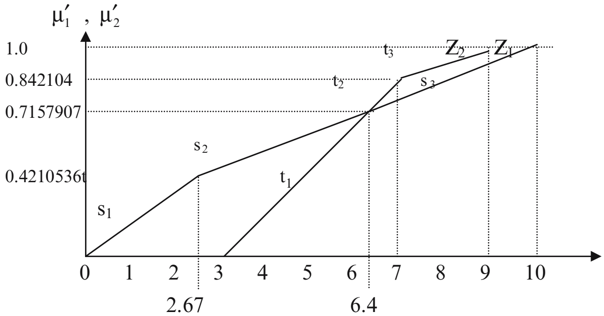
where $\mu_1(z_1)$ and $\mu_2(z_2)$ are depicted in Figure 8(a).

Referring to Remark 1, we know $\mu_1(z_1=8)$ is a convex-type point in z_1 and $\mu_2(z_2=5)$ is a convex-type point in z_2. Then, based on Remark 3, the mapping points can be computed by $b_1' = \mu_1^{-1}(\mu_2(z_2=5)) = 2.67$ and $b_2' = \mu_2^{-1}(\mu_1(z_1=8)) = 7$.

In reference to Remark 4, we have two converted functions below: [as shown in Figure 8(b)]



(a) Two functions μ_1 and μ_2 in Example 2.



(b) Two functions μ_1' and μ_2' in Example 2.

Figure 8. Membership functions

$$\mu_1'(z_1) = s_1(z_1 - 0) + \frac{s_2 - s_1}{2}(|z_1 - 2.67| + z_1 - 2.67) + \frac{s_2 - s_1}{2}(|z_1 - 6.4| + z_1 - 6.4) \qquad (14)$$

$$\mu_2'(z_2) = t_1(z_2 - 3) + \frac{t_2 - t_1}{2}(|z_2 - 6.4| + z_2 - 6.4) + \frac{s_2 - s_1}{2}(|z_2 - 7|| + z_2 - 7) \qquad (15)$$

where $\mu_1(0) = \mu_1'(0) = 0$, $\mu_1(10) = \mu_1'(10) = 2.67s_1 + 3.73s_2 + 3.6s_3 = 1$, $\mu_2(3) = \mu_2'(3) = 0$, $\mu_2(9) = \mu_2'(9) = 3.4t_1 + 0.6t_2 + 2t_3 = 1$,

$$\frac{\mu_1(2.67)}{\mu_2(5)} = \frac{\mu_1'(2.67)}{\mu_2'(5)} = \frac{2.67s_1}{2t_1} = 1$$

$$\frac{\mu_1(6.4)}{\mu_2(6.4)} = \frac{\mu_1'(6.4)}{\mu_2'(6.4)} = \frac{2.67s_1 + 3.73s_2}{3.4t_1} = 1$$

$$\frac{\mu_1(8)}{\mu_2(7)} = \frac{\mu_1'(8)}{\mu_2'(7)} = \frac{2.67s_1 + 3.73s_2 + 1.6s_3}{3.4t_1 + 0.6t_2} = 1$$

$s_1>s_2>s_3>0$, and $t_1>t_2>t_3>0$.

After computed, the slopes of two converted concave functions are $s_1 = 0.157698$, $s_2 = 0.079018$, $s_3 = 0.078947$, $t_1 = 0.210526$, $t_2 = 0.210526$, and $t_3 = 0.078947$. Hence, Example 2 can be transformed into

Example 2:
Maximize λ'
subject to $\lambda' \le \mu_1'(z_1)$, $\lambda' \le \mu_2'(z_2)$, $z_1 = -x_1 + 2x_2$, $z_2 = 2x_1 + x_2$,

$-x_1 + 3x_2 \le 6$, $x_1 + 3x_2 \le 12$, $4x_1 + 3x_2 \le 30$, $3x_1 + x_2 \le 15$, $x_1, x_2 \ge 0$

where $\mu_1'(z_1)$ and $\mu_2'(z_2)$ are expressed in Eqs. (15) and (16), respectively.

$$\mu_1'(z_1) = 0.157698z_1 - \frac{0.07968}{2}(|z_1 - 2.67| + z_1 - 2.67) + \frac{0.000071}{2}(|z_1 - 6.4| + z_1 - 6.4) \quad (16)$$

$$\mu_2'(z_2) = 0.210526(z_2 - 3) + \frac{0}{2}(|z_2 - 6.4| + z_2 - 6.4) + \frac{0.131579}{2}(|z_2 - 7|| + z_2 - 7) \quad (17)$$

Assume that R is the universal set of real numbers, D is an arbitrary domain, and R^n denotes n-dimensional Euclidean space. For any real-valued function u: $D\text{->}R$, the image of D by u is denoted by $u(D)$; i.e., $u(D)=\{u(d)|\ d\in D\}$. Then Inuiguchi et al. (1990) proved that there exists a strictly increasing and objective function g: $u(D)$->$u'(D)$ such that $u'(d)$ = $g(u(d))$ for any d belonging to D, where u:D->R and u':D->R.

Define an r-level set of u:D->R by $[u]_r = \{d \in D \mid f(d) \geq r\}$ where $r \in R$. Inuiguchi et al. (1990) proved that the solution maximizing a function u is equal to the solution maximizing a function u' in any restricted domain when $\{[u]_r \mid r \in u(D)\} = \{[u']_{r'} \mid r' \in u'(D)\}$ and $[u']_{r'}$ is a objective function of $[u]_r$. Accordingly, we have the following proposition.

PROPOSITION 3.

The optimal solution of P1 is the same as that of P2; P1 and P2 are given below in which μ_1, μ_2, μ_1', and μ_2' are specified in Eqs. (3)−(13):

P1	P2
Maximize λ	Maximize λ'
Subject to $\lambda \leq \mu_1(f(X))$	Subject to $\lambda' \leq \mu_1'(f(X))$
$\lambda \leq \mu_2(f(X))$	$\lambda' \leq \mu_2'(f(X))$
$a_1 = b_1 \leq f(X) \leq a_3 = b_3$	$a_1 = b_1 \leq f(X) \leq a_3 = b_3$
$f(X) \in F$ (F is a feasible set).	$f(X) \in F$ (F is a feasible set).

Proof.

For an $f(X)$ in the restricted domain $[a_1, a_3]$ or $[b_1, b_3]$, we have

$$\mu_1(a_1) = \mu_1'(a_1),\ \mu_1(a_3) = \mu_1'(a_3),\ \mu_2(b_1) = \mu_2'(b_1),\ \mu_2(b_3) = \mu_2'(b_3)$$

$$\frac{\mu_1(b_2')}{\mu_2(b_2')} = \frac{\mu_1'(b_2')}{\mu_2'(b_2')},\ \frac{\mu_1(a_2)}{\mu_2(a_2)} = \frac{\mu_1'(a_2)}{\mu_2'(a_2)},\ \frac{\mu_1(b_2)}{\mu_2(b_2)} = \frac{\mu_1'(b_2)}{\mu_2'(b_2)},$$

and $t_3 > 0$, $s_3 > s_4 > s_5 > 0$.

Since $\{\mu_1'(f(X)), \mu_2'(f(X))\}$ is the strictly increasing and bijective function of $\{\mu_1(f(X)), \mu_2(f(X))\}$, $\max_{f(X)} \min\{\mu_1(f(X)), \mu_2(f(X))\}$ is equivalent to $\max_{f(X)} \min\{\mu_1'(f(X)), \mu_2'(f(X))\}$. Therefore, the optimal solution of P1 is the same as the optimal solution of P2.

Take Example 2, for instance. Solve Example 2 by using Proposition 4, discussed in next paragraph, the obtained solution z_1 = 3.525553, z_2 = 5.321128, x_1 = 1.423341, x_2 = 2.474447. The optimal solution of Example 2 is the same as the optimal solution of Example 2. □

PROPOSITION 4.

By referring to Proposition 1, consider an FMODM problem as follows:

Maximize λ
subject to $\lambda \leq \mu_i(z_i), X \in F$ (a feasible set),

where

$\mu_i(z_i) = \mu_i(a_1) + s_1(z_i(X) - a_1) + \sum_{k=2}^{m-1} \frac{s_k - s_{k-1}}{2}(|z_i(X) - a_k| + z_i(X) - a_k)$ is a concave function (i.e., $s_k - s_{k-1} < 0$ for $k = 2, 3, ..., m-1$).

This FMOP problem can then be reformulated as follows:

$$\text{Maximize } \lambda \tag{17}$$

subject to $\lambda \leq \mu_i(z_i)$

$$\mu_i(z_i) = \mu_i(a_1) + s_1(z_i(X) - a_1) + \sum_{k=2}^{m-1}(s_k - s_{k-1})(z_i(X) - a_k + \sum_{\ell=1}^{k-1} d_\ell)$$

$z_i(X) - a_{m-1} + \sum_{\ell=2}^{m-1} d_{\ell-1} \geq 0, 0 \leq d_{\ell-1} \leq a_\ell - a_{\ell-1}$ for all ℓ, $\ell = 2, 3, ..., m-1$,

$X \in F$ (a feasible set).

Proof.

By referring to Li (1996), a GP problem {Maximize $w = \sum_{k=2}^{m-1}(|z_i(X) - a_k| + z_i(X) - a_k)$ subject to: $z_i(X) \geq 0$ and $0 < a_2 < a_3 < \ldots < a_{m-1}$.} is equivalent to

{Maximize $w = 2\sum_{k=2}^{m-1}(z_i(X) - a_k + r_{k-1})$ subject to: $z_i(X) - a_k + r_{k-1} \geq 0$ for $k = 2, 3, \ldots, m-1$, $r_{k-1} \geq 0$, $x_i \geq 0$, where r_{k-1} are deviation variables} (18)

Eq. (20) implies if $z_i(X) < a_k$ then at optimal solution $r_{k-1} = a_k - z_i(X)$; if $z_i(X) \geq a_k$ then at optimal solution $r_{k-1} = 0$. Substitute r_{k-1} by $\sum_{\ell=1}^{k-1} d_\ell$, where d_ℓ is within the bounds as $0 \leq d_\ell \leq a_{\ell+1} - a_\ell$, Equation (20) then becomes

Maximize

$$w = 2\sum_{k=2}^{m-1}\left(z_i(X) - a_k + \sum_{\ell=1}^{k-1} d_\ell\right) \quad (19)$$

subject to $z_i(X) + d_1 \geq a_2$

$z_i(X) + d_1 + d_2 \geq a_3$

$\vdots \qquad \vdots$

$z_i(X) + d_1 + d_2 + \ldots + d_{m-2} \geq a_{m-1}$

$0 \leq d_\ell \leq a_{\ell+1} - a_\ell$ for $\ell = 1, 2, \ldots, m-2$ and $z_i(X) \geq 0$.

Since $a_{\ell+1} - a_\ell \geq d_\ell$ for all ℓ, it is clear that

$$z_i(X) \geq a_{m-1} - \sum_{\ell=1}^{m-2} d_\ell \geq a_{m-2} - \sum_{\ell=1}^{m-3} d_\ell \geq \ldots \geq a_3 - d_1 - d_2 \geq a_2 - d_1 \geq 0\,.$$

The first (m-3) constraints in Model (21) therefore are covered by the (m-2)-th constraint in Model (21). Proposition 4 is then proven. □

Consider the following example as depicted in Figure 9(c):

Example 3:

Maximize

$$z = 1.5x - \frac{0.5}{3}(|x-2| + x - 2) - \frac{0.75}{2}(|x-3| + x\text{-}3)$$

subject to $x \leq 2.5$.

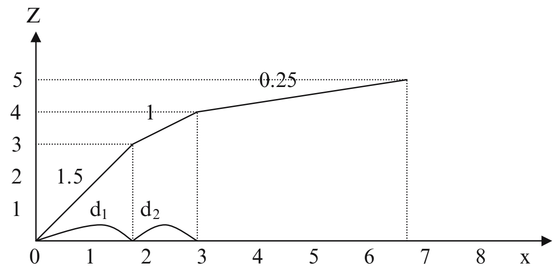


Figure 9. Z function

Referring to Proposition 4, Example 3 can be linearized as
Example 3:
Maximize $z = 1.5x - 0.5(x - 2 + d_1) - 0.75(x - 3 + d_1 + d_2)$
subject to $x + d_1 + d_2 \geq 3$, $0 \leq d_1 \leq 2$, $0 \leq d_2 \leq 1$, and $x \leq 2.5$.
After running on LINGO, we obtained $z = 3.5$, $x = 2.5$, $d_1 = 0$, and $d_2 = 0.5$.

4. SOLUTION ALGORITHMS

From the basis of Proposition 1 to Proposition 4, we propose Algorithm 1 for treating a quasi-concave FMODM problem. From the basis of Algorithm 1, Algorithm 2 is developed for solving an FMODM problem with more general nonconcave membership functions.

Algorithm 1 (Solve a quasi-concave FMOP problem):

Step 1. Use Proposition 1 to express each piece-wise membership function as

$$\mu_i(z_i) = \left(\begin{array}{l} \mu_i(a_{i1}) + s_{i1}(z_i(X) - a_{i1}) + \\ \sum_{k=2}^{M(i)-1} \frac{s_{ik} - s_{ik-1}}{2} (| z_i(X) - a_{ik} | + z_i(X) - a_{ik}) \end{array} \right)$$

where a_{ik}, $k = 1, 2, ..., m$ are the break points of $\mu_i(z_i)$, s_{ik}, $k = 1, 2, ..., m-1$, are the slopes of line segments between a_{ik} and $a_{i,k+1}$, and $i = 1, 2, .., n$.

Step 2. Use Remark 1 to find the convex-type break points and Remark 3 to obtain the corresponding mapping points.

Step 3. Use Remark 4 to specify the converted concave membership functions.

Step 4. Use Equations (11)−(13) to compute the slopes of the converted concave membership functions.

Step 5. Use Proposition 4 to linearize the converted functions and then solve it by LP techniques.

Based on the above discussion, for tackling more general non concave FMODM problems the following remark is presented.

Remark 5 (Model the union of some quasi-concave membership functions). Any piece-wise membership function can be regarded as the union of some quasi-concave membership functions. Take Figure 1(d) for instance, $\mu_i(z_i)$ can be regarded as the union of three quasi-concave functions $\mu_{i1}(a_1 \le z_i \le a_2)$, $\mu_{i2}(a_2 \le z_i \le a_3)$, and $\mu_{i3}(a_3 \le z_i \le a_4)$.

The program of

Maximize λ
subject to $\lambda \le \mu_i(z_i)$ for $i = 1, 2, \ldots, n$

can be rewritten as the following program by referring to Li & Yu (1991).

Maximize λ
Subject to $\lambda \le \mu_{i1}(z_i) + M\delta_1$, $\lambda \le \mu_{i2}(z_i) + M\delta_2$
$\lambda \le \mu_{i3}(z_i) + M\delta_3$, $\delta_1 + \delta_2 + \delta_3 = 1$,
where M is a big number and $\delta_1, \delta_2, \delta_3$ are zero-one variables.

From the basis of Remark 5, Algorithm 2 for solving a general non concave FMOP problem is described as follows.

Algorithm 2 (Solve a general nonconcave FMOP problem):

Step 0. Convert the piece-wise membership functions into the union of some quasi-concave membership functions by adding some zero−one variables.

Steps 1−5 are the same as in Algorithm 1.

5. NUMERICAL EXAMPLES

How to solve Example 1 using Algorithm 1 is illustrated below:

Step 1 . Use Proposition 1 to represent $\mu_1(z_1)$ and $\mu_2(z_2)$ as following Equations [as depicted in Figure 2(a)].

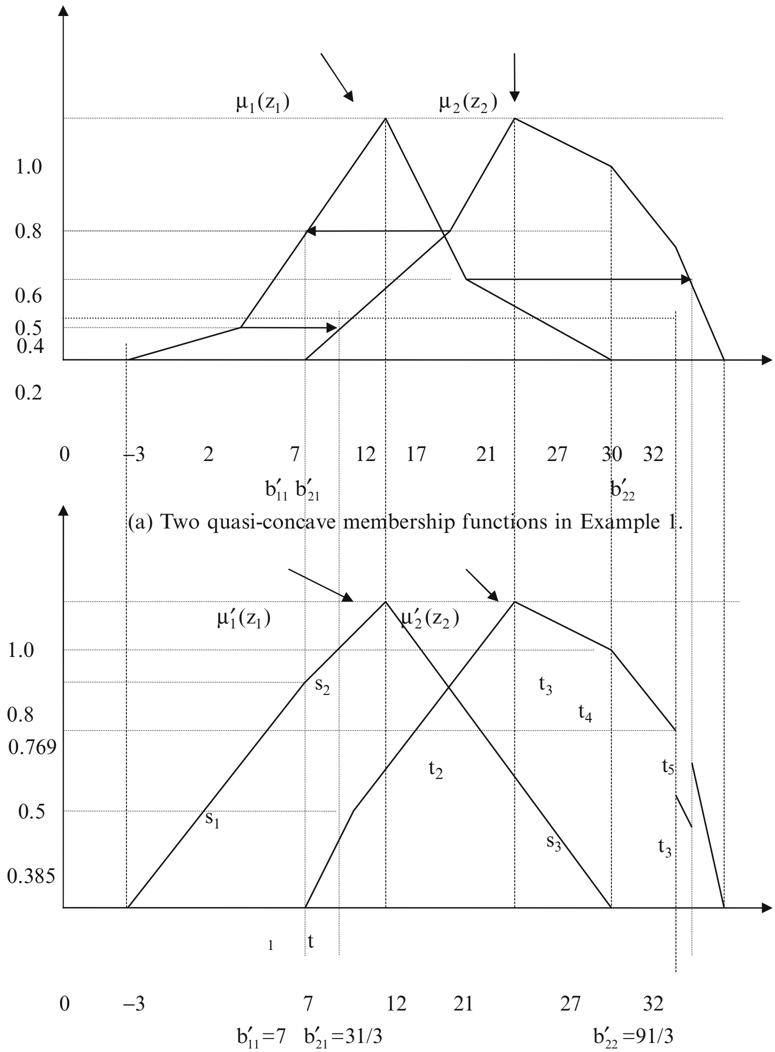
$$\mu_1(z_1) = \begin{pmatrix} 0.04(z_1+3) + 0.02(|z_1-2| + z_1 - 2) - 0.1(|z_1-12| + \\ z_1 - 12) + 0.04(|z_1-17| + z_1 - 17) \end{pmatrix}$$

$$\mu_2(z_2) = \begin{pmatrix} 0.06(z_2-7) + 0.02(|z_2-17| + z_2 - 17) - 0.0665(|z_2-21| + \\ z_2 - 21) - 0.03335(|z_2-27| + z_2 - 27) - 0.075(|z_2-30| + z_2 - 30) \end{pmatrix}$$

Step 2. Use Remark 1 to find convex-type points and Remark 3 to calculate their corresponding mapping points as follows [as depicted in Figure 10(a)]:

$$b_{11} = \mu_1^{-1}[\mu_2(17)] = 7\,, \qquad b_{21} = \mu_2^{-1}[\mu_1(2)] = \frac{31}{3}$$

and $b_{22} = \mu_2^{-1}[\mu_1(17)] = \dfrac{91}{3}$.



(a) Two quasi-concave membership functions in Example 1.

(b) Two converted concave membership functions in Algorithm 1.

Figure 10. Membership functions

Step 3. Use Remark 4 to specify the converted concave membership functions $\mu_1'(z_1)$ and $\mu_2'(z_2)$, as shown in Figure 10(b).

$$\mu_1'(z_1) = \begin{pmatrix} s_1(z_1+3) + \frac{s_2 - s_1}{2}(|z_1 - 7| + z_1 - 7) + \\ \frac{s_3 - s_2}{2}(|z_1 - 12| + z_1 - 12) \end{pmatrix} \quad (20)$$

$$\mu_2'(z_2) = \begin{pmatrix} t_1(z_2 - 7) + \frac{t_2 - t_1}{2}(|z_2 - \frac{31}{3}| + z_2 - \frac{31}{3}) + \frac{t_3 - t_2}{2}(|z_2 - 21| + \\ z_2 - 21) + \frac{t_4 - t_3}{2}(|z_2 - 27| + z_2 - 27) + \frac{t_5 - t_4}{2}(|z_2 - 30| + \\ z_2 - 30) + \frac{t_6 - t_5}{2}(|z_2 - \frac{91}{3}| + z_2 - \frac{91}{3}) \end{pmatrix} \quad (21)$$

Step 4. Use Eqs. (11)−(13) to compute the slopes s_i and t_j, $i = 1, 2, 3$ and $j = 1, 2, ..., 6$ in (20) and (21). Then

$$\mu_1(12) = \mu_1'(12) = 10s_1 + 5s_2 = 1$$

$$\mu_1(27) = \mu_1'(27) = 10s_1 + 5s_2 + 15s_3 = 0$$

$$\mu_2(21) = \mu_2'(21) = \frac{26}{3}t_1 + \frac{16}{3}t_2 = 1$$

$$\mu_2(32) = \mu_2'(32) = \frac{10}{3}t_1 + \frac{32}{3}t_2 + 6t_3 + 3t_4 + \frac{1}{3}t_5 + \frac{5}{3}t_6 = 0$$

$$\frac{\mu_1(2)}{\mu_2(\frac{31}{3})} = \frac{\mu_1'(2)}{\mu_2'(\frac{31}{3})} = \frac{5s_1}{\frac{10}{3}t_1} = 1$$

$$\frac{\mu_1(7)}{\mu_2(17)} = \frac{\mu_1'(7)}{\mu_2'(17)} = \frac{10s_1}{\frac{10}{3}t_1 + \frac{20}{3}t_2} = 1$$

$$\frac{\mu_1(14)}{\mu_2(27)}=\frac{\mu_1'(14)}{\mu_2'(27)}=\frac{10s_1+5s_2+2s_3}{\frac{10}{3}t_1+\frac{32}{3}t_2+6t_3}=1$$

$$\frac{\mu_1(16)}{\mu_2(30)}=\frac{\mu_1'(16)}{\mu_2'(30)}=\frac{10s_1+5s_2+4s_3}{\frac{10}{3}t_1+\frac{32}{3}t_2+6t_3+3t_4}=1$$

$$\frac{\mu_1(17)}{\mu_2(\frac{91}{3})}=\frac{\mu_1'(17)}{\mu_2'(\frac{91}{3})}=\frac{10s_1+5s_2+5s_3}{\frac{10}{3}t_1+\frac{32}{3}t_2+6t_3+3t_4+\frac{1}{3}t_5}=1$$

$s_1>s_2>s_3$, $t_1>t_2>t_3>t_4>t_5>t_6$

After running on the LINGO (2005), the found solutions are $s_1 = 0.077$, $s_2 = 0.046$, $s_3 = -0.067$, $t_1 = 0.11539$, $t_2 = 0.058$, $t_3 = -0.022$, $t_4 = -0.044$, $t_5 = -0.2$, and $t_6 = -0.4$. Therefore, we have

$$\mu_1'(z_1)=0.077(z_1+3)-0.015(|z_1-7|+z_1-7)-0.056(|z_1-12|+z_1-12)$$

$$\mu_2'(z_2)=\begin{pmatrix}0.115(z_2-7)-0.029(|z_2-\frac{31}{3}|+z_2-\frac{31}{3})-0.0399(|z_2-21|+\\ z_2-21)-0.011(|z_2-27|+z_2-27)-0.078(|z_2-30|+z_2-30)-\\ 0.1(|z_2-\frac{91}{3}|+z_2-\frac{91}{3})\end{pmatrix}$$

Step 5. Use Proposition 4 to linearize the converted functions and then solve it by linear programming techniques.

Based on Proposition 4, the linearized program is described below:

FMODM Model 7 (Yu and Li Method for Example 1)
Maximize λ'

subject to $\lambda' \le \mu_1'(Z_1)=0.067Z_1-0.031d_1-0.113d_2+1.804$

$$\lambda' \le \lambda' \le \mu_2'(Z_2) = \begin{pmatrix} -0.4Z_2 - 0.058d_3 - 0.0798d_4 - 0.022d_5 - \\ 0.156d_6 - 0.2d_7 + 12.808 \end{pmatrix}$$

$$z_1 - 7 + d_1 \ge 0,\ z_1 - 12 + d_2 \ge 0, \quad z_2 - \frac{31}{3} + d_3 \ge 0$$

$$z_2 - 21 + d_4 \ge 0,\ z_2 - 27 + d_5 \ge 0, \quad z_2 - 30 + d_6 \ge 0$$

$$z_2 - \frac{91}{3} + d_7 \ge 0, \quad z_1 = -x_1 + 2x_2, \quad z_2 = 2x_1 + x_2, \quad -x_1 + 3x_2 \le 21$$

$$x_1 + 3x_2 \le 27, \quad 4x_1 + 3x_2 \le 45, \quad 3x_1 + x_2 \le 30, \quad x_1,\ x_2 \ge 0$$

By solving on the LINGO, we obtained $x_1 = 5.62$, $x_2 = 7.13$, $z_1 = 8.64$, and $z_2 = 18.36$ in which is exactly the optimal solution of Example 1. Table 1 summarizes the efficiency comparison between Algorithm 1 and conventional FMODM methods for solving Example 1.

Table 1. Efficiency Comparison for Solving Example 1

FMOP Models	Required zero−one variables	Required extra constraints	Required subproblems	Required LP computation	Required point calculation
Narasimhan's and Hannan's methods	Cannot treat Example 1				
FMODM Model 3 (Inuiguchi et al. Method)	3	2	0	1	5
FMODM Model 4 (Yang et al. Method)	3	9	0	1	10
FMODM Model 6 (Nakamura Method)	0	9	8	8	0
FMODM Model 7 (Yu and Li Method)	0	2	0	1	3

Now let us consider the following piece-wise nonconcave FMODM problem.

Example 4
Maximize λ

subject to $\lambda \le \mu_1(z_i)$

$$= \begin{pmatrix} 0.04(z_1+3)+0.02(|z_1-2|+z_1-2)-0.1(|z_1-12|+z_1-12)+ \\ 0.04(|z_1-17|+z_1-17)+0.04(|z_1-27|+z_1-27)+ \\ 0.02(|z_2-42|+z_2-42) \end{pmatrix}$$

$\lambda \le \mu_2(z_2) =$

$$\begin{pmatrix} 0.06(z_2-7)+0.02(|z_2-17|+z_2-17)-0.0665(|z_2-21|+ \\ z_2-21)-0.03335(|z_2-27|+z_2-27)-0.075(|z_2-30|+ \\ z_2-30)+0.145(|z_2-32|+z_2-32)+0.03(|z_2-37|+z_2-37) \end{pmatrix}$$

where $\mu_1(z_i)$ and $\mu_2(z_2)$ are non concave functions as depicted in Figure 11(a).

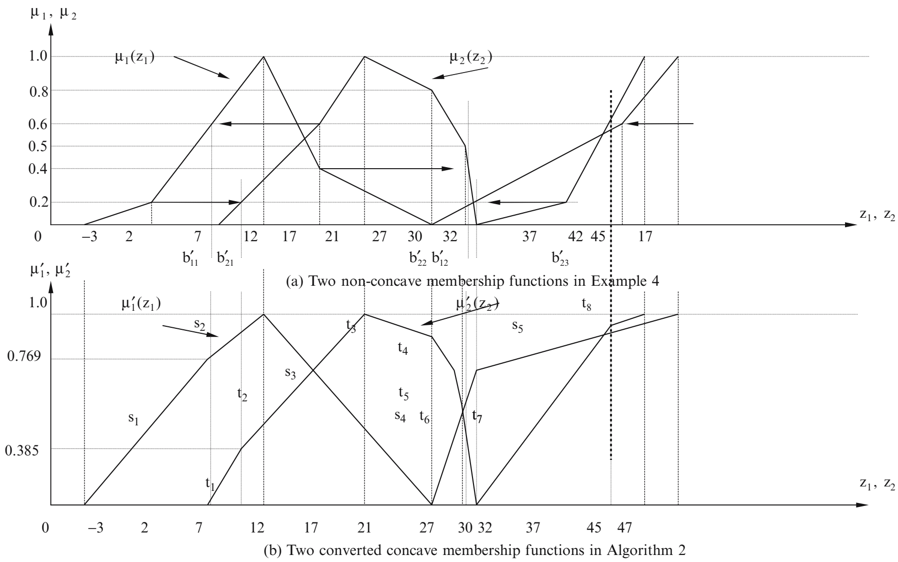


Figure 11. Membership Function

By referring to Algorithm 2, the following steps are illustrated to solve Example 4.

Step 0. Here $\mu_1(z_1)$ can be regarded as the union of two quasi−concave functions $\mu_{11}(-3 \le z_1 \le 27)$ and $\mu_{12}(27 \le z_1 \le 47)$. $\mu_2(z_2)$ can be regarded as the union of two quasi−concave functions $\mu_{21}(7 \le z_2 \le 32)$ and $\mu_{22}(32 \le z_2 \le 45)$.

In reference to Remark 1, Example 2 can reformulated as

$$\text{Maximize} \quad \lambda \tag{23}$$

subject to $\lambda \le \mu_{11}(-3 \le z_1 \le 27) + M\delta_1$

$\lambda \le \mu_{12}(27 \le z_1 \le 47) + M(1-\delta_1)$

$\lambda \le \mu_{21}(7 \le z_2 \le 32) + M\delta_2$

$\lambda \le \mu_{22}(32 \le z_2 \le 45) + M(1-\delta_2)$

where M is a big number and δ_1, δ_2 are 0-1 variables.

Step 1. Employ Proposition 1 to represent $\mu_{11}(-3 \le z_1 \le 27)$, $\mu_{12}(27 \le z_1 \le 47)$, $\mu_{21}(7 \le z_2 \le 32)$ and $\mu_{22}(32 \le z_2 \le 45)$ as follows:

$$\mu_{11}(-3 \le z_1 \le 27) = \begin{pmatrix} 0.04(z_1+3) + 0.02(|z_1-2|+z_1-2) - \\ 0.1(|z_1-12|+z_1-12) + 0.04(|z_1-17|+z_1-17) \end{pmatrix}$$

$$\mu_{12}(27 \le z_1 \le 47) = 0.04(z_1-27) + 0.02(|z_1-42|+z_1-42)$$

$$\mu_{21}(7 \le z_2 \le 32) = \begin{pmatrix} 0.06(z_2-7) + 0.02(|z_2-17|+z_2-17) - \\ 0.0665(|z_2-21|+z_2-21) - 0.03335(|z_2-27|+z_2-27) - \\ 0.075(|z_2-30|+z_2-30) \end{pmatrix}$$

$$\mu_{22}(32 \le z_2 \le 45) = 0.04(z_2-32) + 0.03(|z_2-37|+z_2-37)$$

Step 2. Based on Remarks 1 and 3, after finding the convex-type point, then the mapping points can be obtained by following equations:

$$b'_{11} = \mu_1^{-1}[\mu_2(17)] = 7, \; b'_{12} = \mu_1^{-1}[\mu_2(37)] = 32$$

$b'_{21} = \mu_2^{-1}[\mu_1(2)] = \frac{31}{3}$, $b'_{22} = \mu_2^{-1}[\mu_1(17)] = \frac{91}{3}$
and $b'_{23} = \mu_2^{-1}[\mu_1(42)] = 41$.

Step 3. Using Remark 4 to specify the converted functions

$\mu'_{11}(z_1)$, $\mu'_{12}(z_1)$, $\mu'_{21}(z_2)$, and $\mu'_{22}(z_2)$ as shown in Figure 11(b), respectively:

$$\mu'_{11}(z_1) = \begin{pmatrix} s_1(z_1+3) + \frac{s_2-s_1}{2}(|z_1-7|+z_1-7)+ \\ \frac{s_3-s_2}{2}(|z_1-12|+z_1-12) \end{pmatrix} \tag{24}$$

$$\mu'_{12}(z_1) = s_4(z_1-27) + \frac{s_5-s_4}{2}(|z_1-32|+z_1-32) \tag{25}$$

$$\mu'_{21}(z_2) = \begin{pmatrix} t_1(z_2-7) + \frac{t_2-t_1}{2}(|z_2-\frac{31}{3}|+z_2-\frac{31}{3}) + \frac{t_3-t_2}{2}(|z_2-21|+ \\ z_2-21) + \frac{t_4-t_3}{2}(|z_2-27|+z_2-27) + \frac{t_5-t_4}{2}(|z_2-30|+ \\ z_2-30) + \frac{t_6-t_5}{2}(|z_2-\frac{91}{3}|+z_2-\frac{91}{3}) \end{pmatrix} \tag{26}$$

$$\mu'_{22}(z_2) = t_7(z_2-32) + \frac{t_8-t_7}{2}(|z_2-41|+z_2-41) \tag{27}$$

Step 4. In reference to Eqs. (10)−12), the slopes s_i and t_j, i = 1, 2, .., 5 and j = 1, 2, ..., 8, in Eqs. (24)−(27) can be computed by solving the following equations:

$s_1>s_2>s_3$, $s_4>s_5$, $t_1>t_2>t_3>t_4>t_5>t_6$, $t_7>t_8$

$$\mu_{11}(12) = \mu'_{11}(12) = 10s_1 + 5s_2 = 1$$
$$\mu_{11}(27) = \mu'_{11}(27) = 10s_1 + 5s_2 + 15s_3 = 0$$

$$\mu_{12}(47) = \mu'_{12}(47) = 5s_4 + 15s_5 = 1$$

$$\mu_{21}(21) = \mu'_{21}(21) = \frac{26}{3}t_1 + \frac{16}{3}t_2 = 1$$

$$\mu_{21}(32) = \mu'_{21}(32) = \frac{10}{3}t_1 + \frac{32}{3}t_2 + 6t_3 + 3t_4 + \frac{1}{3}t_5 + \frac{5}{3}t_6 = 0$$

$$\mu_{22}(45) = \mu'_{22}(45) = 9t_7 + 4t_8 = 1$$

$$\frac{\mu_{11}(2)}{\mu_{21}\left(\frac{31}{3}\right)} = \frac{\mu'_{11}(2)}{\mu'_{21}\left(\frac{31}{3}\right)} = \frac{5s_1}{\frac{10}{3}t_1} = 1$$

$$\frac{\mu_{11}(7)}{\mu_{21}(17)} = \frac{\mu'_{11}(7)}{\mu'_{21}(17)} = \frac{10s_1}{\frac{10}{3}t_1 + \frac{20}{3}t_2} = 1$$

$$\frac{\mu_{11}(14)}{\mu_{21}(27)} = \frac{\mu'_{11}(14)}{\mu'_{21}(27)} = \frac{10s_1 + 5s_2 + 2s_3}{\frac{10}{3}t_1 + \frac{32}{3}t_2 + 6t_3} = 1$$

$$\frac{\mu_{11}(16)}{\mu_{21}(30)} = \frac{\mu'_{11}(16)}{\mu'_{21}(30)} = \frac{10s_1 + 5s_2 + 4s_3}{\frac{10}{3}t_1 + \frac{32}{3}t_2 + 6t_3 + 3t_4} = 1$$

$$\frac{\mu_{11}(17)}{\mu_{21}\left(\frac{91}{3}\right)} = \frac{\mu'_{11}(17)}{\mu'_{21}\left(\frac{91}{3}\right)} = \frac{10s_1 + 5s_2 + 5s_3}{\frac{10}{3}t_1 + \frac{32}{3}t_2 + 6t_3 + 3t_4 + \frac{1}{3}t_5} = 1$$

$$\frac{\mu_{12}(32)}{\mu_{22}(37)} = \frac{\mu'_{12}(32)}{\mu'_{22}(37)} = \frac{10s_1 + 5s_2 + 15s_3 + 5s_4}{\frac{10}{3}t_1 + \frac{32}{3}t_2 + 6t_3 + 3t_4 + \frac{1}{3}t_5 + \frac{5}{3}t_6 + 5t_7} = 1$$

$$\frac{\mu_{12}(42)}{\mu_{22}(41)} = \frac{\mu'_{12}(42)}{\mu'_{22}(41)} = \frac{10s_1 + 5s_2 + 15s_3 + 5s_4 + 10s_5}{\frac{10}{3}t_1 + \frac{32}{3}t_2 + 6t_3 + 3t_4 + \frac{1}{3}t_5 + \frac{5}{3}t_6 + 9t_7} = 1$$

After computing on the LINGO, the found solutions are s_1 = 0.077, s_2 = 0.046, s_3 = −0.067, s_4 = 0.091, s_5 = 0.0364, t_1 = 0.11539, t_2 = 0.058, t_3 = −0.022, t_4 = −0.044, t_5 = −0.2, t_6 = –0.4, t_7 = 0.091, and t_8 = 0.0455.

Therefore, the program (23) becomes

Maximize λ'

$$\text{subject to } \lambda' \le \begin{pmatrix} 0.077(Z_1+3)-0.0154(|Z_1-7|+Z_1-7)- \\ 0.056576(|Z_1-12|+Z_1-12)+M\delta_1 \end{pmatrix}$$

$$\lambda' \le 0.158(Z_1-27)-0.0273(|Z_1-42|+Z_1-42)+M(1-\delta_1)$$

$$\lambda' \le \begin{pmatrix} 0.11539(Z_2-7)-0.029(|Z_2-\frac{31}{3}|+Z_2-\frac{31}{3})-0.0399(|Z_2-21|+ \\ Z_2-21)-0.011(|Z_2-27|+Z_2-27)-0.078(|Z_2-30|+Z_2-30)- \\ 0.1(|Z_2-\frac{91}{3}|+Z_2-\frac{91}{3})+M\delta_2 \end{pmatrix}$$

$$\lambda' \le 0.49(Z_2-32)-0.023(|Z_2-41|+Z_2-41)+M(1-\delta_2),$$

where M is a big number and δ_1, δ_2 are zero−one variables.

Step 5. Employing Proposition 4, the above problem can then be linearized below:

Maximize λ'

Subject to $\lambda' \le 0.067Z_1 - 0.031d_1 - 0.113d_2 + 1.804 + M\delta_1$,

$\lambda' \le 0.103Z_1 - 0.0556d_3 - 1.973 + M(1-\delta_1)$,

$\lambda' \le 0.4Z_2 - 0.058d_4 - 0.0798d_5 - 0.022d_6 - 0.156d_7 - 0.2d_8 + 12.808 + M\delta_2$,

$\lambda' \le 0.444Z_2 - 0.046d_9 - 13.794 + M(1-\delta_2)$,

$$z_1-7+d_1 \ge 0,\quad z_1-12+d_2 \ge 0,\quad z_1-42+d_3 \ge 0$$

$$z_2-(31/3)+d_4 \ge 0,\quad z_2-21+d_5 \ge 0,\quad z_2-27+d_6 \ge 0$$

$$z_2-30+d_7 \ge 0,\quad z_2-(91/3)+d_8 \ge 0,\quad z_2-41+d_9 \ge 0$$

$$z_1 = -x_1 + 2x_2, \quad z_2 = 2x_1 + x_2, \quad -x_1 + 3x_2 \le 21, \quad x_1 + 3x_2 \le 27$$

$$4x_1 + 3x_2 \le 45, \quad 3x_1 + x_2 \le 30, \quad x_1, x_2 \ge 0$$

After running on LINGO, the obtained solutions are $x_1 = 5.62$, $x_2 = 7.13$, $z_1 = 8.64$ and $z_2 = 18.36$. This is exactly the optimal solution of Example 2. Table 2 displays the comparisons between traditional FMODM methods and Algorithm 2.

Table 2. Efficiency Comparison for Solving Example 2

FMODM models	Required zero-one variables	Required extra constraints	Required subproblems	Required LP computation	Required point calculation
Narasimhan's, Hannan's and Inuiguchi et al. methods	Cannot treat Example 2.				
Yang et al. Method	9	26	0	1	14
Nakamura Method	0	26	26	26	0
Yu and Li Method	2	4	0	1	5

6. CONCLUDING REMARKS

With the remarkable advance of computer technology in the last two decades, how to solve real-world FMODM problems to obtain the best acceptable solution with an efficient algorithm has received considerable attention among scientists, engineers, and managers. Since the powerful advantage of a computerized system strongly depends on the availability and effectiveness of a mathematical formulation, in reality decision makers widely use either stochastic or fuzzy programming to treat uncertain MODM problems. Stochastic uncertainty is related to environment data such as consumer demand and inflows, whereas fuzzy uncertainty concerns the use of approximate values by the decision maker when setting objective values (Abdelaziz et al., 2004). Consequently, after Zimmermann first introduced conventional LP and multi-objective LP into fuzzy set theory, various methods using LP were developed to tackle the FMODM problems.

However, membership functions, whenever used in LP optimization, as reported in literature are generally restricted to linear, triangular, or trapezoid functions. This main restriction has excluded many important domains of application. Many empirical studies (Biswal, 1997; Hannan, 1981; Mjelde, 1983; Nakamura, 1984; Narasimhan, 1980; Yang et al., 1991;) report that real-world membership functions in the engineering, physical, business, social, and management fields are not pure linear, triangular, concave, or convex shapes but rather than more general non concave curves. Therefore, this chapter has been devoted to solving a quasi-concave or more general nonconcave FMODM problem. Comparing with conventional FMODM methods, the proposed method can directly solve a quasi-concave FMODM problem by using standard LP techniques. Moreover, there is no requirement to add extra zero−one variables or to divide the original problem into several sub-problems for solving a quasi-concave FMODM problem. Without a tiresome solution process, the proposed method can be extended to solve more general nonconcave FMOP problems by adding less number of zero−one variables. Numerical examples are employed to illustrate the practicability and applicability of the proposed method.

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INTERACTIVE FUZZY MULTI-OBJECTIVE STOCHASTIC LINEAR PROGRAMMING

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Abstract: Two major approaches to deal with randomness or ambiguity involved in mathematical programming problems have been developed. They are stochastic programming approaches and fuzzy programming approaches. In this chapter, we focus on multiobjective linear programming problems with random variable coefficients in objective functions and/or constraints. Using several stochastic models such as an expectation optimization model, a variance minimization model, a probability maximization model, and a fractile criterion optimization model in chance constrained programming, the stochastic programming problems are transformed into deterministic ones. As a fusion of stochastic approaches and fuzzy ones, after determining the fuzzy goals of the decision maker, several interactive fuzzy satisfying methods to derive a satisfying solution for the decision maker by updating the reference membership levels are presented.

Key words: Fuzzy mathematical programming, multi-criteria analysis, linear programming, stochastic programming, interactive programming

1. INTRODUCTION

In actual decision-making situations, we must often make a decision on the basis of vague information or uncertain data. For such decision-making problems involving uncertainty, there exist two typical approaches: stochastic programming and fuzzy programming.

Stochastic programming, as an optimization method on the basis of the probability theory, has been developing in various ways (Stancu-Minasian, 1984; 1990), including a two-stage problem by G.B. Dantzig (1955), and chance constrained programming by A. Charnes and W.W. Cooper (1959). In particular, for multi-objective stochastic linear programming problems, I.M. Stancu-Minasian (1984, 1990) considered the minimum risk approach, while J.P. Leclercq (1982) and Teghem Jr. et al. (1986) proposed interactive methods.

On the other hand, fuzzy mathematical programming representing the vagueness in decision-making situations by fuzzy concepts has been studied by many researchers (Lai and Hwang, 1992; Rommelfanger, 1996; Sakawa, 1993). Fuzzy multiobjective linear programming, first proposed by H.-J. Zimmermann (1978), has also been developed by numerous researchers, and an increasing number of successful applications have been introduced (Delgado et al., 1994; Kacprzyk and Orlovski, 1987; Lai and Hwang, 1994; Luhandjula, 1987; Sakawa et al. 1987; Sakawa, 2001; 1993; 2000; Slowinski, 1998; Slowinski and Teghem, 1990; Verdegay and Delgado, 1989; Zimmermann, 1987).

As a hybrid of the stochastic approach and the fuzzy one, Wang et al. (Wang and Qiao, 1993) and Luhandjula et al. (Luhandjula, 1996; Luhandjula and Gupta, 1996) considered mathematical programming problems with fuzzy random variables (Kwakernaak, 1778; Puri, 1986), and Liu and Iwamura (1998) discussed chance constrained programming involving fuzzy parameters. In particular, Hulsurkar et al. (1997) applied fuzzy programming to multi-objective stochastic linear programming problems. Unfortunately, however, in their method, since membership functions for the objective functions are supposed to be aggregated by a minimum operator or a product operator, optimal solutions that sufficiently reflect the decision maker's preference may not be obtained.

Under these circumstances, in this chapter, we focus on multi-objective linear programming problems with random variable coefficients in objective functions and/or constraints. Through the use of several stochastic models, including an expectation optimization model, a variance minimization model, a probability maximization model, and a fractile criterion optimization model together with chance constrained programming techniques, the stochastic programming problems are transformed into deterministic ones. Assuming that the decision maker has a fuzzy goal for each objective function, having determined the fuzzy goals of the decision maker, we present several interactive fuzzy satisfying methods to derive a satisfying solution for the decision maker by updating the reference membership levels. As an illustrative numerical example, a multi-objective linear programming problem involving random variable coefficients for the probability maximization model is provided to demonstrate the feasibility of the proposed method.

2. MULTI-OBJECTIVE LINEAR PROGRAMMING PROBLEMS WITH RANDOM VARIABLE COEFFICIENTS

Throughout this chapter, we deal with multi−objective linear programming problems where coefficients in objective functions and right-hand side constants of constraints are assumed to be random. Such multi-objective linear programming problems involving random variable coefficients are formally formulated as:

$$\left.\begin{array}{ll} \text{Minimize} & z_1(x,\omega) = c_1(\omega)x \\ & \vdots \\ \text{Minimize} & z_k(x,\omega) = c_k(\omega)x \\ \text{subject to} & Ax \le b(\omega) \\ & x \ge 0 \end{array}\right\} \quad (1)$$

where $\boldsymbol{x}$ is an n-dimensional decision variable column vector and A is an $m \times n$ coefficient matrix.

It should be noted that $c_l(\omega), l = 1,\dots,k$ are n-dimensional random variable row vectors with finite mean $\bar{\boldsymbol{c}}_l$ and finite covariance matrix $V_l = (v^l_{jh}) = (\text{Cov}[c_{lj}(\omega), c_{lh}(\omega)]),\ j = 1,\dots,n,\ h = 1,\dots,n$ and $b_i(\omega)$, $i = 1,\dots,m$ are random variables with finite mean $\bar{b}_i$, which are independent of each other, and the distribution function of each of them is also assumed to be continuous and increasing.

Multi-objective linear programming problems with random variable coefficients are said to be multi-objective stochastic linear programming ones, which are often seen in actual decision-making situations. For example, consider a production planning problem to optimize the gross profit and production cost simultaneously under the condition that unit profits of the products, unit production costs of them, and the maximal amounts of the resources depend on seasonal factors or market prices. Such a production planning problem can be formulated as a multi-objective programming problem with random variable coefficients expressed by Eq. (1).

Since the formulated problem Eq.(1) contains random variable coefficients, definitions and solution methods for ordinary mathematical programming problems cannot be directly applied. Consequently, we deal with the constraints in Eq. (1) as chance constrained conditions (Charnes and Cooper, 1959), which mean that the constraints need to be satisfied with a certain probability (satisfying level) and over. Namely, replacing the constraints in Eq. (1) by chance constrained conditions with satisfying levels β_i, $i = 1,\ldots,m$, Eq. (1) can be converted as

$$\left.\begin{array}{ll} \text{Minimize} & z_1(x,\omega) = c_1(\omega)x \\ & \vdots \\ \text{Minimize} & z_k(x,\omega) = c_k(\omega)x \\ \text{subject to} & \Pr\left[a_1 x \leq b_1(\omega)\right] \geq \beta_1 \\ & \vdots \\ & \Pr\left[a_m x \leq b_m(\omega)\right] \geq \beta_m \\ & x \geq 0 \end{array}\right\} \tag{2}$$

where $\boldsymbol{a}_i$ is the ith row vector of A and $b_i(\omega)$ is the ith element of $b(\omega)$.

Denoting continuous and increasing distribution functions of random variables $b_i(\omega)$, $i = 1,\ldots,m$ by $F_i(r) = \Pr[b_i(\omega) \leq r]$, the ith constraint in Eq.(2) can be rewritten as:

$$\begin{aligned} \Pr\left[a_i x \leq b_i(\omega)\right] \geq \beta_i &\Leftrightarrow 1 - \Pr\left[b_i(\omega) \leq a_i x\right] \geq \beta_i \\ &\Leftrightarrow 1 - F_i(a_i x) \geq \beta_i \\ &\Leftrightarrow F_i(a_i x) \leq 1 - \beta_i \\ &\Leftrightarrow a_i x \leq F_i^{-1}(1 - \beta_i) \end{aligned} \tag{3}$$

Letting $\hat{b}_i = F_i^{-1}(1 - \beta_i)$, Eq.(2) can be transformed into the following equivalent problem:

$$\left.\begin{array}{ll} \text{Minimize} & z_1(x,\omega) = c_1(\omega)x \\ & \vdots \\ \text{Minimize} & z_k(x,\omega) = c_k(\omega)x \\ \text{subject to} & A\,x \leq \hat{b} \\ & x \geq 0 \end{array}\right\} \tag{4}$$

where $\hat{b} = (\hat{b}_1,...,\hat{b}_m)^T$. In the following section, for notational convenience, the feasible region of Eq. (4) is denoted by X.

For the multi-objective chance constrained programming problem Eq. (4), several stochastic models such as an expectation optimization model, a variance minimization model, a probability maximization model, and a fractile criterion model have been proposed depending on the concern of the decision maker.

3. EXPECTATION OPTIMIZATION MODEL

In this section, we state the expectation optimization model for multi-objective chance constrained programming problems (Sakawa and Kato, 2002; Sakawa et al. 2003b), where the decision maker aims to optimize the expectation of each objective function represented as a random variable in Eq. (4).

Substituting the objective functions $z_l(\boldsymbol{x},\omega) = \boldsymbol{c}_l(\omega)\boldsymbol{x}$, $l = 1,...,k$ in Eq. (4) for their expectations, the problem can be converted as

$$\left.\begin{array}{ll} \text{Minimize} & \mathrm{E}\left[z_1(x,\omega)\right] \\ & \vdots \\ \text{Minimize} & \mathrm{E}\left[z_k(x,\omega)\right] \\ \text{subject to} & \mathrm{A}\,x \le \hat{b} \\ & x \ge 0 \end{array}\right\} \tag{5}$$

Letting $\bar{c}_l = \mathrm{E}[c_l(\omega)]$, $\bar{z}_l(x) = \mathrm{E}[z_l(x,\omega)]$ can be expressed as

$$\bar{z}_l(\boldsymbol{x}) = \bar{\boldsymbol{c}}_l\boldsymbol{x}. \tag{6}$$

Then, Eq. (5) can be reduced to the following ordinary multi-objective linear programming problem:

$$\left.\begin{array}{ll} \text{Minimize} & \bar{c}_1 x \\ & \vdots \\ \text{Minimize} & \bar{c}_k x \\ \text{subject to} & \mathrm{A}\,x \le \hat{b} \\ & x \ge 0. \end{array}\right\} \tag{7}$$

In order to consider the imprecise nature of the decision maker's judgments for each objective function

$$\bar{z}_l(x) = \bar{c}_l x$$

in Eq. (7), if we introduce the fuzzy goals such as "$\bar{z}_l(\boldsymbol{x})$ should be substantially less than or equal to a certain value," Eq. (7) can be rewritten as

$$\underset{x \in X}{\text{Maximize}} \quad (\mu_1(\bar{z}_1(x)), \ldots, \mu_k(\bar{z}_k(x))) \tag{8}$$

where $\mu_l(\cdot)$ is a membership function to quantify a fuzzy goal for the lth objective function in Eq. (7). To be more specific, if the decision maker feels that $\bar{z}_l(\boldsymbol{x})$ should be greater than or equal to at least $\bar{z}_{l,0}$ and that $\bar{z}_l(x) \geq \bar{z}_{l,1}(> \bar{z}_{l,0})$ is satisfactory, the shape of a typical membership function is shown in Figure 1.

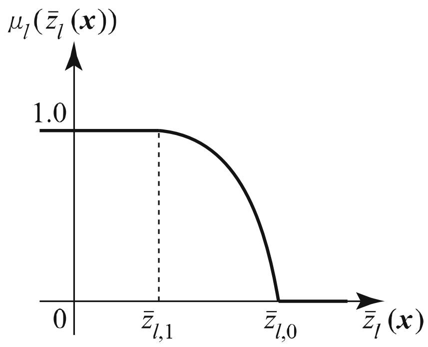

Figure 1. An example of a membership function $\mu_l(\bar{z}_l(x))$

Since Eq. (8) is regarded as a fuzzy multi-objective decision-making problem, there rarely exists a complete optimal solution that simultaneously optimizes all objective functions. As a reasonable solution concept for the fuzzy multi-objective decision-making problem, M. Sakawa et al. (Sakawa and Yano, 1985; 1990; Sakawa et al., 1987; Sakawa, 1993) defined M-Pareto optimality on the basis of membership function values by directly extending the Pareto optimality in the ordinary multi-objective programming problem.

DEFINITION 1 (M-PARETO OPTIMAL SOLUTION)

$\boldsymbol{x}^* \in X$ is said to be an M-Pareto optimal solution if and only if there does not exist another $\boldsymbol{x} \in X$ such that $\mu_l(\bar{z}_l(\boldsymbol{x})) \geq \mu_l(\bar{z}_l(\boldsymbol{x}^*))$ for $l = 1,\ldots,k$ and $\mu_j(\bar{z}_j(x)) > \mu_j(\bar{z}_j(x^*))$ for at least one $j \in \{1,\ldots,k\}$.

Introducing an aggregation function $\mu_D(\boldsymbol{x})$ for k membership functions in Eq. (8), the problem can be rewritten as:

$$\left.\begin{array}{ll}\text{Minimize} & \mu_D(\boldsymbol{x}) \\ \text{subject to} & \boldsymbol{x} \in X\end{array}\right\} \tag{9}$$

The aggregation function $\mu_D(\boldsymbol{x})$ represents the degree of satisfaction or preference of the decision maker for the whole of k fuzzy goals.

Following the conventional fuzzy approaches, as aggregation functions, Hulsurkar et al. (1997) adopted the minimum operator of Bellman and Zadeh (1970) defined by

$$\mu_D(\mathbf{x}) = \min_{l=1,\ldots,k} \{\mu_l(\bar{z}_l(\mathbf{x}))\} \cdot$$

and the product operator of Zimmermann (1978) defined by

$$\mu_D(x) = \prod_{l=1}^{k} \mu_l(\bar{z}_l(x)) \,.$$

However, it should be emphasized here that such approaches are preferable only when the decision maker feels that the minimum operator or the product operator is appropriate. In other words, in general decision situations, the decision maker does not always use the minimum operator or the product operator when combining the fuzzy goals. Probably the most crucial problem in Eq. (9) is the identification of an appropriate aggregation function that well represents the decision maker's fuzzy preferences. If $\mu_D(\boldsymbol{x})$ can be explicitly identified, then Eq. (9) reduces to a standard mathematical programming problem. However, this rarely happens, and as an alternative, an interaction with the decision maker is necessary for finding a satisfying solution to Eq. (9).

In an interactive fuzzy satisfying method, to generate a candidate for a satisfying solution that is also M-Pareto optimal, the decision maker is asked to specify the aspiration levels of achievement for the membership values of all membership functions, called the reference membership levels (Sakawa and Yano, 1985; 1989; 1990; Sakawa et al., 1987; Sakawa, 1993).

For the decision maker's reference membership levels $\bar{\mu}_l, l = 1,\dots,k$, the corresponding M-Pareto optimal solution, which is nearest to the requirements in the minimax sense or better than that if the reference membership levels are attainable, is obtained by solving the following minimax problem:

$$\left.\begin{array}{ll} \text{Minimize} & \max_{l=1,\dots,k} \{\bar{\mu}_l - \mu_l(\bar{z}_l(x))\} \\ \text{subject to} & x \in X \end{array}\right\}. \tag{10}$$

By introducing the auxiliary variable v, this problem can be equivalently transformed as

$$\left.\begin{array}{ll} \text{Minimize} & v \\ \text{subject to} & \bar{\mu}_1 - \mu_1(\bar{z}_1(x)) \le v \\ & \vdots \\ & \bar{\mu}_k - \mu_k(\bar{z}_k(x)) \le v \\ & x \in X \end{array}\right\}. \tag{11}$$

If the value of v is fixed to v^*, Eq. (11) can be reduced to a linear programming problem. Therefore, we can find an optimal solution $(\boldsymbol{x}^*, v^*)$ corresponding to v^* by the bisection method based on the simplex method.

Following the preceding discussions, we can now construct the interactive algorithm in order to derive the satisfying solution for the decision maker from the M-Pareto optimal solution set. The steps marked with an asterisk involve interaction with the decision maker.

Interactive fuzzy satisfying method for expectation optimization model

Step 1. Calculate the individual minimum $\bar{z}_l^{min}$ and maximum $\bar{z}_l^{max}$ of $\mathrm{E}[z_l(\boldsymbol{x},\omega)] = \bar{z}_l(\boldsymbol{x}), l = 1,\dots,k$ under the chance constrained conditions with satisfying levels $\beta_i, i = 1,\dots,m$ by solving the following linear programming problems:

$$\underset{x \in X}{\text{minimize}} \quad \bar{z}_l(x) = \bar{c}_l x, \; l = 1,\dots,k \tag{12}$$

$$\underset{x \in X}{\text{maximize}} \quad \bar{z}_l(x) = \bar{c}_l x, \; l = 1,\dots,k \tag{13}$$

Step 2. Ask the decision maker to determine membership functions $\mu_l(\bar{z}_l(\boldsymbol{x}))$ for objective functions in Eq. (7).

Step 3. Ask the decision maker to set the initial reference membership levels $\bar{\mu}_l = 1, l = 1,\ldots,k$.

Step 4. Solve the following minimax problem

$$\left.\begin{array}{ll} \text{Minimize} & \max\limits_{l=1,\ldots,k} \{\bar{\mu}_l - \mu_l(\bar{z}_l(x))\} \\ \text{subject to} & x \in X \end{array}\right\} \tag{14}$$

corresponding to the reference membership levels $\bar{\mu}_l, l = 1,\ldots,k$. To be more specific, after calculating the optimal value v^* to the problem

$$\left.\begin{array}{ll} \text{Minimize} & \boldsymbol{v} \\ \text{subject to} & \bar{z}_1(x) \le \mu_1^{-1}(\bar{\mu}_1 - v) \\ & \quad\vdots \\ & \bar{z}_k(x) \le \mu_k^{-1}(\bar{\mu}_k - v) \\ & x \in X \end{array}\right\} \tag{15}$$

by the bisection method and phase one of the two-phase simplex method, solve the linear programming problem

$$\left.\begin{array}{ll} \text{Minimize} & \bar{z}_1(x) \\ \text{subject to} & \bar{z}_2(x) \le \mu_2^{-1}(\bar{\mu}_2 - v^*) \\ & \quad\vdots \\ & \bar{z}_k(x) \le \mu_k^{-1}(\bar{\mu}_k - v^*) \\ & x \in X \end{array}\right\} \tag{16}$$

where $z_1(x,\omega)$ is supposed to be the most important to the decision maker.

For the obtained $\boldsymbol{x}^*$, if there are inactive constraints in the first $(k-1)$ constraints, replace $\bar{\mu}_l$ for inactive constraints with $\mu_l(\bar{z}_l(\boldsymbol{x}^*)) + v^*$ and resolve the corresponding problem. Furthermore, if the obtained $\boldsymbol{x}^*$ is not unique, perform the M-Pareto optimality test.

Step 5. The decision maker is supplied with the corresponding M-Pareto optimal solution and the trade-off rates between the membership functions. If the decision maker is satisfied with the current membership function values of the M-Pareto optimal solution, stop. Otherwise, ask the decision maker to update the reference membership levels $\mu_l,\ l = 1,\ldots,k$ by considering the current membership function values $\mu_l\left(\bar{z}_l\left(x^*\right)\right)$ together with the trade-off rates $-\partial\mu_1 / \partial\mu_l$, $l = 2,\ldots,k$ and return to step 4. Here, the trade-off rates are expressed as

$$-\frac{\partial\mu_1(\bar{z}_1(x))}{\partial\mu_l(\bar{z}_l(x))} = \pi_l \cdot \frac{\mu_1'(\bar{z}_1(x^*))}{\mu_l'(\bar{z}_l(x^*))}, \quad l = 2,\ldots,k$$

where $\pi_l, l = 2,\ldots,k$ are simplex multipliers in Eq. (16).

Since the trade-off rates $-\partial\mu_1 / \partial\mu_l, l = 2,\ldots,k$ in step 5 indicate the decrement of value of a membership function μ_1 with a unit increment of value of a membership function μ_l, they are employed to estimate the local shape of $(\mu_1(\bar{z}_1(\boldsymbol{x}^*)),\ldots,\mu_k(\bar{z}_k(\boldsymbol{x})))$ around $\boldsymbol{x}^*$.

Here it should be stressed to the decision maker that any improvement of one membership function can be achieved only at the expense of at least one of the other membership functions.

For further details along this line, the readers might refer to the corresponding papers (Sakawa et al., 2000; 2003b).

4. VARIANCE MINIMIZATION MODEL

Since objective functions regarded as random variables in Eq. (4) are reduced to their expectations in the expectation−optimization model, the requirement of the decision maker for risk is not reflected in the obtained solution. From this viewpoint, in this section, we consider the variance minimization model for multi-objective chance constrained programming problems (Sakawa et al., 2002). In the model, we substitute the minimization of variances of objective functions for the minimization of objective functions in Eq. (4). Then, the problem can be rewritten as

$$\left.\begin{array}{ll} \text{Minimize} & z_1'(x) = \mathrm{Var}\left[z_1(x,\omega)\right] = x^T V_1 x \\ & \vdots \\ \text{Minimize} & z_k'(x) = \mathrm{Var}\left[z_k(x,\omega)\right] = x^T V_k x \\ \text{subject to} & A x \le \hat{b} \\ & x \ge 0 \end{array}\right\} \quad (17)$$

Using the variance minimization model, the obtained solution might be too bad in the sense of the expectation of objective functions, while it accomplishes the minimization in the sense of the variance. In order to take the requirement of the decision maker for expectations of objective functions into account, we consider the following revised variance minimization model incorporating constraints that the expectation of each objective function, $\bar{z}_l(x) = \bar{c}_l x$ must be less than or equal to a certain permissible level $\gamma_l, l = 1,\ldots,k$.

$$\left.\begin{array}{ll} \text{Minimize} & z_1'(x) = \text{Var}\left[z_1(x,\omega)\right] = x^T V_1 x \\ & \vdots \\ \text{Minimize} & z_k'(x) = \text{Var}\left[z_k(x,\omega)\right] = x^T V_k x \\ \text{subject to} & A x \le \hat{b} \\ & \bar{C} x \le \gamma \\ & x \ge 0 \end{array}\right\} \quad (18)$$

where $\bar{C} = (\bar{c}_1^T,\ldots,\bar{c}_k^T)^T$ and $\gamma = (\gamma_1,\ldots,\gamma_k)^T$, and we denote the feasible region of Eq. (18) by X'.

In order to consider the imprecise nature of the decision maker's judgments for each objective function in Eq. (18), if we introduce the fuzzy goals such as "$z_l'(x)$ should be substantially less than or equal to a certain value," the problem Eq. (18) can be rewritten as

$$\underset{x \in X}{\text{Maximize}} \quad (\mu_1(z_1'(x)),\ldots,\mu_k(z_k'(x))) \quad (19)$$

where $\mu_l(\cdot)$ is a membership function to quantify a fuzzy goal for the lth objective function in Eq. (18) shown in Figure 2.

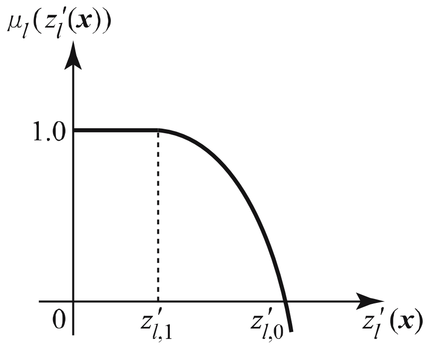

Figure 2. An example of a membership function $\mu_l(z_l'(\boldsymbol{x}))$

In order to derive a satisfying solution for the decision maker from the M-Pareto optimal solution set, Sakawa et al. (Sakawa and Yano, 1985; 1990; Sakawa et al., 1987; Sakawa, 1993) proposed an interactive fuzzy satisfying method such that the decision maker interactively updates the aspiration levels of achievement for the membership values of all membership functions, called the reference membership levels, until he is satisfied.

We now summarize the interactive algorithm.

Interactive fuzzy satisfying method for variance minimization model

Step 1. Ask the decision maker to specify the satisfying levels β_i, $i = 1, \ldots, m$ for each of the constraints in Eq. (1).

Step 2. After calculating the individual minimum $\bar{z}_l^{min}$ and maximum $\bar{z}_l^{max}$ of $\mathrm{E}[z_l(x, \omega)] = \bar{z}_l(x), l = 1, \ldots, k$ under the chance constrained conditions, ask the decision maker to specify permissible levels γ_l, $l = 1, \ldots, k$ for objective functions.

Step 3. Calculate the individual minimum $z_{l,min}^{\prime}$ of $z_l^{\prime}(\boldsymbol{x}), l = 1, \ldots, k$ in Eq. (18) by solving the following quadratic programming problems:

$$\underset{x \in X'}{\text{minimize}} \quad z_l^{\prime}(x) = x^T V_l \, x, \; l = 1, \ldots, k \tag{20}$$

Step 4. Ask the decision maker to determine membership functions $\mu_l(z_l^{\prime}(\boldsymbol{x}))$ for objective functions in (18) on the basis of individual minima $z_{l,min}^{\prime}$.

Step 5. Ask the decision maker to set the initial reference membership levels

$$\bar{\mu}_l = 1, l = 1, \ldots, k .$$

Step 6. Calculate the optimal solution $\boldsymbol{x}^*$ to the augmented minimax problem Eq. (21) corresponding to the current reference membership levels $\bar{\mu}_l, l = 1, \ldots, k$.

$$\underset{x \in X'}{\text{Minimize}} \max_{l=1,\ldots,k} \left[\bar{\mu}_l - \mu_l(z_l^{\prime}(x)) + \rho \sum_{i=1,\ldots,k} (\bar{\mu}_i - \mu_i(z_i^{\prime}(x))) \right] \tag{21}$$

Here, assuming that each membership functions $\mu_l(\cdot), l = 1, \ldots, k$ is nonincreasing and concave, Eq. (21) is a convex programming problem.

Under the assumption, we can solve Eq. (21) by a traditional convex programming technique as the sequential quadratic programming method.

***Step** 7.* The decision maker is supplied with the obtained solution $\boldsymbol{x}^*$. If the decision maker is satisfied with the current membership function values of $\boldsymbol{x}^*$, stop. Otherwise, ask the decision maker to update the reference membership levels $\bar{\mu}_l, l = 1,\ldots,k$ by considering the current membership function values $\mu_l(z_l^t(\boldsymbol{x}^*))$, and return to step 6.

5. PROBABILITY MAXIMIZATION MODEL

In this section, we investigate the probability maximization model for a multi-objective chance constrained programming problem (Sakawa and Kato, 2002; Sakawa et al., 2004), where the decision maker aims to maximize the probability that each objective function represented as a random variable is less than or equal to a certain permissible level in Eq. (4).

Substituting the minimization of the objective functions $z_l(x,\omega) = c_l(\omega)x, l = 1,\ldots,k$ in Eq. (4) for the maximization of the probability that each objective function $z_l(\boldsymbol{x},\omega)$ is less than or equal to a certain permissible level f_l, the problem can be converted as

$$\left.\begin{array}{ll} \text{Maximize} & p_1(x) = \Pr\left[z_1(x,\omega) \le f_1\right] \\ & \vdots \\ \text{Maximize} & p_k(x) = \Pr\left[z_k(x,\omega) \le f_k\right] \\ \text{subject to} & Ax \le \hat{b} \\ & x \ge 0 \end{array}\right\} \tag{22}$$

In order to consider the imprecise nature of the decision maker's judgment for each objective function in Eq. (22), if we introduce the fuzzy goals such as "$p_l(\boldsymbol{x})$ should be substantially greater than or equal to a certain value," Eq. (22) can be rewritten as

$$\underset{\mathbf{x}\in X}{\text{Maximize}} \ (\mu_1(p_1(\mathbf{x})),\ldots,\mu_k(p_k(\mathbf{x}))) \tag{23}$$

where $\mu_l(\cdot)$ is a membership function to quantify a fuzzy goal for the l th objective function in Eq. (22). To be more explicit, if the decision maker feels that $p_l(\boldsymbol{x})$ should be greater than or equal to at least $p_{l,0}$ and $p_l(x) \ge p_{l,1}(> p_{l,0})$ is satisfactory, the shape of a typical membership function is shown in Figure 3.

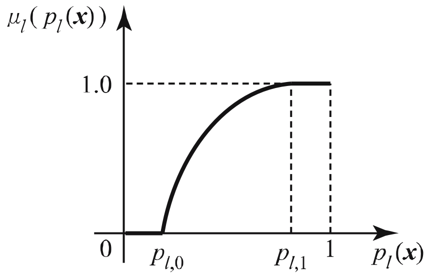

Figure 3. An example of a membership function $\mu_l(p_l(\boldsymbol{x}))$

In an interactive fuzzy satisfying method, to generate a candidate for the satisfying solution that is also M-Pareto optimal, the decision maker is asked to specify the aspiration levels of achievement for the membership values of all membership functions, called the reference membership levels (Sakawa and Yano, 1985; 1989; 1990; Sakawa et al., 1987; Sakawa, 1993).

For the decision maker's reference membership levels $\bar{\mu}_l, l = 1,\ldots,k$, the corresponding M-Pareto optimal solution, which is nearest to the requirements in the minimax sense or better than that if the reference membership levels are attainable, is obtained by solving the following minimax problem:

$$\left.\begin{array}{l} \text{Minimize } \max\limits_{l=1,\ldots,k} \{\bar{\mu}_l - \mu_l(p_l(x))\} \\ \text{subject to } x \in X. \end{array}\right\}. \tag{24}$$

By introducing the auxiliary variable v, this problem can be equivalently transformed as

$$\left.\begin{array}{l} \text{Minimize } v \\ \text{subject to } \bar{\mu}_1 - \mu_l(p_1(x)) \leq v \\ \qquad\qquad\quad \vdots \\ \qquad\qquad \bar{\mu}_k - \mu_k(p_k(x)) \leq v \\ \qquad\qquad x \in X. \end{array}\right\}. \tag{25}$$

Now, let every membership function $\mu_l(\cdot)$ be continuous and strictly increasing. Then, (25) is equivalent to the following problem:

$$\left.\begin{array}{l} \text{Minimize } v \\ \text{subject to } p_1(x) \geq \mu_1^{-1}(\overline{\mu}_1 - v) \\ \qquad\vdots \\ \qquad p_k(x) \geq \mu_k^{-1}(\overline{\mu}_k - v) \\ \qquad x \in X \end{array}\right\}. \tag{26}$$

Since Eq. (26) is a nonconvex, nonlinear programming problem in general, it is difficult to solve it.

Here, in Eq. (1), we assume that $c_l(\omega), l = 1,\ldots,k$ dimensional random variable row vectors expressed by $c_l(\omega) = c_l^1 + t_l(\omega)c_l^2$ where $t_l(\omega)$'s are random variables independent of each other, and $\alpha_l(\omega)$'s are random variables expressed by $\alpha_l(\omega) = \alpha_l^1 + t_l(\omega)\alpha_l^2$, where the corresponding distribution function $T_l(\cdot)$ of each of $t_l(\omega)$, s is assumed to be continuous and strictly increasing.

Supposing that $c_l^2 x + \alpha_l^2 > 0, l = 1,\ldots,k$ for any $\boldsymbol{x} \in X$, from the assumption on distribution functions $T_l(\cdot)$ of random variables $t_l(\omega)$, we can rewrite the objective functions in Eq. (22) as

$$\begin{aligned} \Pr\left[z_l(\mathbf{x},\omega) \leq f_l\right] &= \Pr\left[(\mathbf{c}_l^1 + t_l(\omega)\mathbf{c}_l^2)\mathbf{x} + (\alpha_l^1 + t_l(\omega)\alpha_l^2) \leq f_l\right] \\ &= \Pr\left[(\mathbf{c}_l^2\mathbf{x} + \alpha_l^2)t_l(\omega) + (\mathbf{c}_l^1\mathbf{x} + \alpha_l^1) \leq f_1\right] \\ &= \Pr\left[t_l(\omega) \leq \frac{f_l - (\mathbf{c}_l^1\mathbf{x} + \alpha_l^1)}{(\mathbf{c}_l^2\mathbf{x} + \alpha_l^2)}\right] \\ &= T_l\left(\frac{f_l - \mathbf{c}_l^1\mathbf{x} - \alpha_l^1}{\mathbf{c}_l^2\mathbf{x} + \alpha_l^2}\right). \end{aligned}$$

Hence, Eq. (22) can be transformed into the following ordinary multiobjective programming problem:

$$\left.\begin{array}{l} \text{Maximize } p_1(\mathbf{x}) = T_1\left(\dfrac{f_l - \mathbf{c}_1^1\mathbf{x} - \alpha_1^1}{\mathbf{c}_1^2\mathbf{x} + \alpha_1^2}\right) \\ \qquad\qquad\vdots \\ \text{Maximize } p_k(\mathbf{x}) = T_k\left(\dfrac{f_k - \mathbf{c}_k^1\mathbf{x} - \alpha_k^1}{\mathbf{c}_k^2\mathbf{x} + \alpha_k^2}\right) \\ \text{subject to } \mathbf{x} \in X \end{array}\right\}. \tag{27}$$

In view of

$$p_l(\boldsymbol{x}) = T_l\left(\frac{f_l - \boldsymbol{c}_l^1\boldsymbol{x} - \alpha_l^1}{\boldsymbol{c}_l^2\boldsymbol{x} + \alpha_l^2}\right)$$

and the continuity and strictly increasing property of the distribution function $T_l(\cdot)$, This problem can be equivalently transformed as

$$\left.\begin{array}{ll} \text{Minimize } v & \\ \text{subject to } & \dfrac{f_1 - \mathbf{c}_1^1 x - \alpha_1^1}{\mathbf{c}_1^2 x + \alpha_1^2} \geq T_1^{-1}(\mu_1^{-1}(\overline{\mu}_1 - v)) \\ & \vdots \\ & \dfrac{f_k - \mathbf{c}_k^1 x - \alpha_k^1}{\mathbf{c}_k^2 x + \alpha_k^2} \geq T_k^{-1}(\mu_k^{-1}(\overline{\mu}_k - v)) \\ & \mathbf{x} \in X \end{array}\right\}. \tag{28}$$

It is important to note here that, in this formulation, if the value of v +is fixed, it can be reduced to a set of linear inequalities. Obtaining the optimal solution v^* to the above problem is equivalent to determining the maximum value of v so that there exists an admissible set satisfying the constraints of equations (28). Since v satisfies

$$\overline{\mu}_{max} - \max_{l=1,\ldots,k} \mu_{l,max} \leq v \leq \overline{\mu}_{max} - \min_{l=1,\ldots,k} \mu_{l,min}$$

where

$$\overline{\mu}_{max} = \max_{l=1,\ldots,k} \overline{\mu}_l,\ \mu_{l,max} = \max_{\boldsymbol{x}\in X} \mu_l(p_l(\boldsymbol{x})),\ \mu_{l,min} = \min_{\boldsymbol{x}\in X} \mu_l(p_l(\boldsymbol{x}))$$

we can obtain the minimum value of v by combined use of the bisection method and phase one of linear programming technique.

After calculating v^*, the minimum value of v, we solve the following linear fractional programming problem in order to uniquely determine $\boldsymbol{x}^*$ corresponding to v^*:

$$\left.\begin{array}{ll}\text{Minimize} & \dfrac{c_1^1 x + \alpha_1^1 - \mathrm{f}_1}{c_1^2 x + \alpha_1^2} \\ \text{subject to} & \dfrac{\mathrm{f}_2 - c_2^1 x - \alpha_2^1}{c_2^2 x + \alpha_2^2} \geq \mathrm{T}_2^{-1}(\mu_2^{-1}(\bar{\mu}_2 - \mathrm{v}^*)) \\ & \vdots \\ & \dfrac{\mathrm{f}_\mathrm{k} - c_\mathrm{k}^1 x - \alpha_\mathrm{k}^1}{c_\mathrm{k}^2 x + \alpha_\mathrm{k}^2} \geq \mathrm{T}_\mathrm{k}^{-1}(\mu_\mathrm{k}^{-1}(\bar{\mu}_\mathrm{k} - \mathrm{v}^*)) \\ & x \in \mathrm{X} \end{array}\right\} \qquad (29)$$

where $z_1(\boldsymbol{x},\omega)$ is supposed to be the most important to the decision maker.

Using the Charnes−Cooper transformation (Charnes and Cooper, 1962)

$$s = 1/(c_1^2 x + \alpha_1^2), \quad y = s \cdot x, \quad s > 0 \qquad (30)$$

the linear fractional programming problem Eq. (29) is converted to the following linear programming problem

$$\left.\begin{array}{ll}\text{Minimize} & c_1^1 y + (\alpha_1^1 - \mathrm{f}_1)\cdot \mathrm{s} \\ \text{subject to} & \tau_2 \cdot (c_2^2 y + \alpha_2^2 \cdot \mathrm{s}) + c_2^1 y + (\alpha_2^1 - \mathrm{f}_2)\cdot \mathrm{s} \leq 0 \\ & \vdots \\ & \tau_\mathrm{k} \cdot (c_\mathrm{k}^2 y + \alpha_\mathrm{k}^2 \cdot \mathrm{s}) + c_\mathrm{k}^1 y + (\alpha_\mathrm{k}^1 - \mathrm{f}_\mathrm{k})\cdot \mathrm{s} \leq 0 \\ & \mathrm{A}\, y - \mathrm{s}\cdot \hat{b} \leq \mathbf{0} \\ & c_1^2 y + \alpha_1^2 \cdot \mathrm{s} = 1 \\ & -\mathrm{s} \leq -\delta \\ & y \geq \mathbf{0} \\ & \mathrm{s} \geq 0 \end{array}\right\} \qquad (31)$$

where $\tau_l = T_l^{-1}(\mu_l^{-1}(\bar{\mu}_l - v^*))$, and δ is sufficiently small and positive.

If the optimal solution (y^*, s^*) to Eq. (31) is not unique, the Pareto optimality of $x^* = y^*/s^*$ is not guaranteed. The Pareto optimality of $\boldsymbol{x}^*$ can be tested by solving the following linear programming problem.

$$\left.\begin{aligned} &\text{Maximize} \quad \mathrm{w} = \sum_{\mathrm{l}=1}^{\mathrm{k}} \varepsilon_{\mathrm{l}} \\ &\text{subject to} \quad \mathrm{q}_1(x) - \varepsilon_1 = \frac{\mathrm{q}_1(x^*)}{\mathrm{r}_1(x^*)} \cdot \mathrm{r}_1(x) \\ &\qquad\qquad \vdots \\ &\qquad\qquad \mathrm{q}_{\mathrm{k}}(x) - \varepsilon_{\mathrm{k}} = \frac{\mathrm{q}_{\mathrm{k}}(x^*)}{\mathrm{r}_{\mathrm{k}}(x^*)} \cdot \mathrm{r}_{\mathrm{k}}(x) \\ &\qquad\qquad x \in \mathrm{X}, \ \varepsilon = (\varepsilon_1, \ldots, \varepsilon_{\mathrm{k}})^{\mathrm{T}} \geq 0 \end{aligned}\right\} \tag{32}$$

where

$$q_l(x) = f_l - c_l^1 x - \alpha_l^1, \ r_l(x) = c_l^2 x + \alpha_l^2$$

For the optimal solution to Eq. (32), (a) if $w = 0$, i.e., $\varepsilon_l = 0$ for $l = 1,\ldots,k$, $\boldsymbol{x}^*$ is Pareto optimal. On the other hand, (b) $\mathrm{w} > 0$, i.e., $\varepsilon_l > 0$ for at least one l, $\boldsymbol{x}^*$ is not Pareto optimal. Then, we can find a Pareto optimal solution according to the following algorithm.

Step 1. For the optimal solution $\bar{x}, \bar{\varepsilon}$ to the problem (32), after arbitrarily selecting j such as $\bar{\varepsilon}_j > 0$, solve the following problem:

$$\left.\begin{aligned} &\text{Maximize} \quad \frac{f_j - \mathbf{c}_j^1 \mathbf{x} - \alpha_j^1}{\mathbf{c}_j^2 \mathbf{x} + \alpha_j^2} \\ &\text{subject to} \quad \frac{f_l - \mathbf{c}_l^1 \mathbf{x} - \alpha_l^1}{\mathbf{c}_l^2 \mathbf{x} + \alpha_l^2} = \frac{f_l - \mathbf{c}_l^1 \bar{\mathbf{x}} - \alpha_l^1}{\mathbf{c}_l^2 \bar{\mathbf{x}} + \alpha_l^2}, \ \{l \mid \bar{\varepsilon}_l = 0\} \\ &\qquad\qquad \frac{f_l - \mathbf{c}_l^1 \mathbf{x} - \alpha_l^1}{\mathbf{c}_l^2 \mathbf{x} + \alpha_l^2} \geq \frac{f_l - \mathbf{c}_l^1 \bar{\mathbf{x}} - \alpha_l^1}{\mathbf{c}_l^2 \bar{\mathbf{x}} + \alpha_l^2}, \ \{l \mid \bar{\varepsilon}_l > 0\} \\ &\qquad\qquad \mathbf{x} \in X \end{aligned}\right\} \tag{33}$$

Since the above problem can be converted to a linear programming problem by the Charnes and Cooper transformation (Charnes and Cooper, 1962), we can solve it by the simplex method.

Step 2. To test the Pareto optimality of the optimal solution $\hat{x}$ to Eq. (33), solve the problem Eq. (32) where $\hat{x}$ is substituted for $\boldsymbol{x}^*$.

Step 3. If $w=0$, $\hat{x}$ is Pareto optimal and stop. Otherwise, i.e., if $w>0$, return to step 1 since $\hat{x}$ is not Pareto optimal.

Repeating this process at least $k-1$ iterations, a Pareto optimal solution can be obtained.

The decision maker must either be satisfied with the current Pareto optimal solution or act on this solution by updating the reference membership levels. In order to help the decision maker express a degree of preference, trade-off information between a standing membership function and each of the other membership functions is very useful. Such trade-off information is easily obtainable since it is closely related to the simplex multipliers of Eq. (31).

To derive the trade-off information, define the Lagrange function L for Eq. (31) as follows:

$$\begin{aligned} L(y,s,\pi,\sigma,\omega) = {} & c_1^1 y + (\alpha_1^1 - f_1)\cdot s + \sum_{l=2}^{k} \pi_l [\tau_l \cdot (\boldsymbol{c}_l^2 y + \alpha_l^2 \cdot s) \\ & + \{c_l^1 y + (\alpha_l^1 - f_l)\cdot s\}] + \sum_{i=1}^{m} \sigma_i (a_i y - s\cdot \hat{b}_i) \\ & + \sigma_{m+1}\cdot (c_1^2 y + \alpha_1^2 \cdot s - 1) + \sigma_{m+2}\cdot(-s+\delta) \\ & - \sum_{j=1}^{n} \omega_j \cdot y_j - \omega_{n+1}\cdot s \end{aligned} \tag{34}$$

where π, σ, and ω are simplex multipliers.

Then, the partial derivative of $L(y,s,\pi,\sigma,\omega)$ with respect to τ_l is given as follows.

$$\frac{\partial L(y,s,\pi,\sigma,\omega)}{\partial \tau_l} = \pi_l \cdot (c_l^2 y + \alpha_l^2 \cdot s),\ l = 2,\ldots,k \tag{35}$$

On the other hand, for the optimal solution (y^*, s^*) to Eq. (31) and the corresponding simplex multipliers $(\pi^*, \sigma^*, \omega^*)$, the following equation holds from the Kuhn-Tucker necessity theorem (Sakawa, 1993):

$$L(y^*, s^*, \pi^*, \sigma^*, \omega^*) = c_l^1 y^* + (\alpha_l^1 - f_l) \cdot s^* \tag{36}$$

If the first $(k-1)$ constraints to Eq. (31) are active, τ_l is calculated as follows:

$$\tau_l = -\frac{c_l^1 y^* + (\alpha_l^1 - f_l) \cdot s^*}{c_l^2 y^* + \alpha_l^2 \cdot s^*},\ l = 2, \ldots, k . \tag{37}$$

From Eq. (35), Eq. (36) and Eq. (37), for $l = 2, \ldots, k$, we have

$$-\frac{\partial (c_l^1 y^* + (\alpha_l^1 - f_1) \cdot s^*)}{\partial \left(\dfrac{c_l^1 y^* + (\alpha_l^1 - f_l) \cdot s^*}{c_l^2 y^* + \alpha_l^2 \cdot s^*} \right)} = \pi_l^* \cdot (c_l^2 y^* + \alpha_l^2 \cdot s^*) . \tag{38}$$

By substituting $\boldsymbol{x}^*$ for $\boldsymbol{y}^*$, $\boldsymbol{s}^*$ in Eq. (38), the equation is rewritten as

$$-\frac{\partial \left(\dfrac{f_l - \boldsymbol{c}_l^1 \boldsymbol{x}^* - \alpha_l^1}{\boldsymbol{c}_l^2 \boldsymbol{x}^* + \alpha_l^2} \right)}{\partial \left(\dfrac{f_l - \boldsymbol{c}_l^1 \boldsymbol{x}^* - \alpha_l^1}{\boldsymbol{c}_l^2 \boldsymbol{x}^* + \alpha_l^2} \right)} = \pi_l^* \cdot \frac{\boldsymbol{c}_l^2 \boldsymbol{x}^* + \alpha_l^2}{\boldsymbol{c}_1^2 \boldsymbol{x}^* + \alpha_1^2},\ l = 2, \ldots, k . \tag{39}$$

Using the chain rule, the following relation holds:

$$-\frac{\partial T_l \left(\dfrac{f_l - c_l^1 x^* - \alpha_l^1}{c_l^2 x^* + \alpha_l^2} \right)}{\partial T_l \left(\dfrac{f_l - c_l^1 x^* - \alpha_l^1}{c_l^2 x^* + \alpha_l^2} \right)} = \pi_l^* \cdot \frac{c_l^2 x^* + \alpha_l^2}{c_1^2 x^* + \alpha_1^2} \cdot \frac{T_l' \left(\dfrac{f_l - c_l^1 x^* - \alpha_l^1}{c_l^2 x^* + \alpha_l^2} \right)}{T_l' \left(\dfrac{f_l - c_l^1 x^* - \alpha_l^1}{c_l^2 x^* + \alpha_l^2} \right)} . \tag{40}$$

$$l = 2, \ldots, k .$$

Equivalently,

$$-\frac{\partial p_l(x^*)}{\partial p_l(x^*)} = \pi_l^* \cdot \frac{c_l^2 x^* + \alpha_l^2}{c_l^2 x^* + \alpha_l^2} \cdot \frac{p_l'(x^*)}{p_l'(x^*)}, \; l = 2,\ldots,k . \tag{41}$$

Again, using the chain rule, for $l = 2,\ldots,k$ we have

$$-\frac{\partial \mu_l(p_l(x^*))}{\partial \mu_l(p_l(x^*))} = \pi_1^* \cdot \frac{c_l^2 x^* + \alpha_l^2}{c_l^2 x^* + \alpha_l^2} \cdot \frac{p_l'(x^*)}{p_l'(x^*)} \cdot \frac{\mu_l'(p_1(x^*))}{\mu_l'(p_l(x^*))}. \tag{42}$$

It should be stressed here that in order to obtain the trade-off information from Eq. (42), the first $(k-1)$ constraints in Eq. (31) must be active. Therefore, if there are inactive constraints, it is necessary to replace $\bar{\mu}_l$ for inactive constraints with $\mu_l(p_l(\boldsymbol{x}^*)) + v^*$ and solve the corresponding problem to obtain the simplex multipliers.

Following the preceding discussions, we can now construct the interactive algorithm in order to derive the satisfying solution for the decision maker from the Pareto optimal solution set.

Interactive fuzzy satisfying method for probability maximization model

Step 1. Calculating the individual minimum $\bar{z}_l^{\min}$ and maximum $\bar{z}_l^{\max}$ of $\mathrm{E}[z_l(x,\omega)] = \bar{z}_l(x),\ l = 1,\ldots,k$ under the chance constrained conditions with satisfying levels $\beta_i, i = 1,\ldots,m$.

Step 2. Ask the decision maker to specify permissible levels $f_l, l = 1,\ldots,k$ for objective functions.

Step 3. Calculate the individual minimum $p_{l,\min}$ and maximum $p_{l,\max}$ of $p_l(x), l = 1,\ldots,k$ in the multi-objective probability maximization problem Eq. (27) by solving the following problems:

$$\underset{\boldsymbol{x}\in X}{\text{Minimize}}\; p_l(\boldsymbol{x}) = T_l\left(\frac{f_l - \boldsymbol{c}_l^1 \boldsymbol{x} - \alpha_l^1}{\boldsymbol{c}_l^2 \boldsymbol{x} + \alpha_l^2}\right),\; l = 1,\ldots,k \tag{43}$$

$$\underset{\boldsymbol{x}\in X}{\text{Maximize}}\; p_l(\boldsymbol{x}) = T_l\left(\frac{f_l - \boldsymbol{c}_l^1 \boldsymbol{x} - \alpha_l^1}{\boldsymbol{c}_l^2 \boldsymbol{x} + \alpha_l^2}\right),\; l = 1,\ldots,k \tag{44}$$

Then ask the decision maker to determine membership functions $\mu_l(p_l(\boldsymbol{x}))$ for objective functions in Eq. (27).

Step 4. Ask the decision maker to set the initial reference membership levels

$$\overline{\mu}_l = 1, l = 1,\ldots,k$$

Step 5. In order to obtain the optimal solution $\boldsymbol{x}^*$ to the minimax problem Eq. (24) corresponding to the reference membership levels $\overline{\mu}_l = 1, l = 1,\ldots,k$, after solving Eq. (28) by the bisection method and phase one of the two-phase simplex method, solve the linear programming problem Eq. (31). For the obtained $\boldsymbol{x}^*$, if there are inactive constraints in the first $(k-1)$ constraints, replace $\overline{\mu}_l$ for inactive constraints with $\mu_l(p_l(\boldsymbol{x}^*)) + v^*$ and resolve the corresponding problem. Furthermore, if the obtained $\boldsymbol{x}^*$ is not unique, perform the Pareto optimality test.

Step 6. The decision maker is supplied with the corresponding Pareto optimal solution and the trade-off rates between the membership functions. If the decision maker is satisfied with the current membership function values of the Pareto optimal solution, stop. Otherwise, ask the decision maker to update the reference membership levels $\overline{\mu}_l = 1, l = 1,\ldots,k$ by considering the current membership function values $\mu_l(p_l(\boldsymbol{x}^*))$ together with the trade-off rates $-\partial\mu_1 / \partial\mu_l$, $l = 2,\ldots,k$, and return to step 5.

Since the trade-off rates $-\partial\mu_1 / \partial\mu_l$, $l = 2,\ldots,k$ in Step 6 indicate the decrement of value of a membership function μ_1 with a unit increment of value of a membership function μ_l, they are employed to estimate the local shape of $\mu_1(p_1(\boldsymbol{x}^*)),\ldots,\mu_k(p_k(\boldsymbol{x}^*))$ around $\boldsymbol{x}^*$.

Here, as in the discussion for the expectation−optimization model, it should be also stressed to the decision maker that any improvement of one membership function can be achieved only at the expense of at least one of the other membership functions.

6. FRACTILE CRITERION OPTIMIZATION MODEL

In this section, we discuss a fractile criterion model for the multi-objective chance constrained programming problem (Sakawa et al., 2001), which aims to find the minimal value of multiple objective functions such that the probability of obtaining such a result is greater than or equal to some given thresholds under chance constrained conditions.

Substituting the minimization of the objective functions $z_l(x,\omega)$, $l = 1,\ldots,k$ in Eq. (4) for the minimization of values $f_l,\ l = 1,\ldots,k$ such that the probability of obtaining such result is greater than or equal to some given thresholds α_l under a chance constrained condition, the problem can be converted as

$$\left.\begin{array}{ll} \text{Minimize} & f_1 \\ \text{Minimize} & f_k \\ \text{subject to} & \Pr\left[c_1(\omega)x \le f_1\right] \ge \alpha_1 \\ & \quad\vdots \\ & \Pr\left[c_k(\omega)x \le f_k\right] \ge \alpha_k \\ & Ax \le \hat{b} \\ & x \ge \mathbf{0} \end{array}\right\} \tag{45}$$

where $\alpha_l \in (1/2,\ 1),\ l = 1,\ldots,k$ is assumed to guarantee the convexity of the finally reduced problem.

In Eq. (45), we assume that $\boldsymbol{c}_l(\omega), l = 1,\ldots,k$ are Gaussian random variable vectors. Then, the constraints

$$\Pr[c_l(\omega)x \le f_l] \ge \alpha_l,\ \ l = 1,\ldots,k \tag{46}$$

are transformed as

$$\Pr[c_l(\omega)x \le f_l] \ge \alpha_l \Leftrightarrow \Pr\left[\frac{c_l(\omega)x - \bar{c}_l x}{\sqrt{x^T V_l x}} \le \frac{f_l - \bar{c}_l x}{\sqrt{x^T V_l x}}\right] \ge \alpha_l. \tag{47}$$

Since random variables

$$\frac{\boldsymbol{c}_l(\omega)\boldsymbol{x} - \bar{\boldsymbol{c}}_l \boldsymbol{x}}{\sqrt{\boldsymbol{x}^T V_l \boldsymbol{x}}},\ \ l = 1,\ldots,k \tag{48}$$

in the above conditions, are standard Gaussian random variables with mean 0 and variance 1^2, the conditions Eq. (46) are reduced to the following conditions:

$$\Phi\left(\frac{f_l - \bar{c}_l x}{\sqrt{x^T V_l x}}\right) \geq \alpha_l,\ l = 1,\ldots,k$$

$$\Leftrightarrow \frac{f_l - \bar{c}_l x}{\sqrt{x^T V_l x}} \geq K_{\alpha_l},\ l = 1,\ldots,k$$

$$\Leftrightarrow f_l \geq \bar{c}_l x + K_{\alpha_l} \sqrt{x^T V_l x},\ l = 1,\ldots,k$$

where $\Phi(\cdot)$ is the distribution function of a standard Gaussian random variable and $K_{\alpha_l} = \inf \Phi^{-1}(\alpha_l)$. Based on the above discussion, Eq. (45) can be transformed into the following problem:

$$\left.\begin{array}{ll} \text{Minimize} & f_1 \\ \text{Minimize} & f_k \\ \text{subject to} & \bar{c}_1 x + K_{\alpha_1} \sqrt{x^T V_1 x} \leq f_1 \\ & \vdots \\ & \bar{c}_k x + K_{\alpha_k} \sqrt{x^T V_k x} \leq f_k \\ & A x \leq \hat{b} \\ & x \geq \mathbf{0} \end{array}\right\}. \qquad (49)$$

Equivalently, Eq. (49) can be rewritten as

$$\left.\begin{array}{ll} \text{Minimize} & f_1(x) = \bar{c}_1 x + K_{\alpha_1} \sqrt{x^T V_1 x} \\ \text{minimize} & f_k(x) = \bar{c}_k x + K_{\alpha_k} \sqrt{x^T V_k x} \\ \text{subject to} & A x \leq \hat{b} \\ x \geq 0 & \end{array}\right\}. \qquad (50)$$

Furthermore, each of the objective functions in Eq. (50) is convex since each of $K_{\alpha_l}, l = 1,\ldots,k$ is positive from $\alpha_l \in (1/2, 1)$. Therefore, Eq. (50) is a multiobjective convex programming problem. In the following discussion, for notational convenience, the feasible region of Eq. (50) is denoted by *X*.

In order to consider the imprecise nature of the decision maker's judgments for each objective function

$$f_l(x) = \bar{c}_l x + K_{\alpha_l} \sqrt{x^T V_l x}$$

in Eq. (50), if we introduce the fuzzy goals such as "$f_l(\boldsymbol{x})$ should be substantially less than or equal to a certain value", (50) can be rewritten as:

$$\underset{\boldsymbol{x} \in X}{\text{Maximize}} (\mu_1(f_1(\boldsymbol{x})),\ldots,\mu_k(f_k(\boldsymbol{x}))) \tag{51}$$

where $\mu_l(\cdot)$ is a membership function to quantify a fuzzy goal for the *l* th objective function in Eq. (50) and it is assumed to be concave. The shape of a typical membership function is shown in Figure 4.

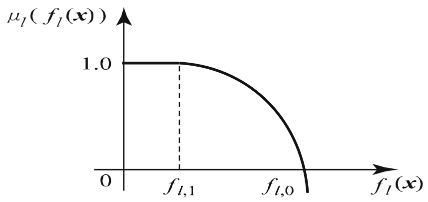

Figure 4. An example of a membership function $\mu_l(f_l(x))$

For the decision maker's reference membership levels $\bar{\mu}_l, l = 1,\ldots,k$, the corresponding M-Pareto optimal solution, which is nearest to the requirements in the minimax sense or better than that if the reference membership levels are attainable, is obtained by solving the following minimax problem:

$$\left.\begin{array}{l} \text{Minimize} \max\limits_{l=1,\ldots,k} \{\bar{\mu}_1 - \mu_l(f_l(\mathbf{x}))\} \\ \text{subject to } \mathbf{x} \in X \end{array}\right\} \tag{52}$$

By introducing the auxiliary variable *v*, this problem can be equivalently transformed as

$$\left.\begin{array}{l} \text{Minimize } v \\ \text{subject to } \bar{\mu}_1 - \mu_1(f_1(\mathbf{x})) \le v \\ \qquad\vdots \\ \qquad \bar{\mu}_k - \mu_k(f_k(\mathbf{x})) \le v \\ \\ \qquad \mathbf{x} \in X. \end{array}\right\} \tag{53}$$

If the optimal solution (x^*, v^*) to Eq. (53) is not unique, the M-Pareto optimality of x^* is not guaranteed. In order to avoid the above situation, we consider the following augmented minimax problem

$$\left.\begin{array}{l} \text{Minimize } v \\ \text{subject to } \bar{\mu}_1 - \mu_1(f_1(\mathbf{x})) + \rho \sum_{i=1}^{k} (\bar{\mu}_i - \mu_i(f_i(\mathbf{x}))) \le v \\ \qquad\vdots \\ \qquad \bar{\mu}_k - \mu_k(f_k(\mathbf{x})) + \rho \sum_{i=1}^{k} (\bar{\mu}_i - \mu_i(f_i(\mathbf{x}))) \le v \\ \\ \qquad \mathbf{x} \in X \end{array}\right\} \tag{54}$$

where ρ is a sufficiently small positive number. It should be noted that the augmented minimax problem (54) is a convex programming problem under the assumption that each of the membership functions $\mu_l(\cdot), l = 1,\ldots,k$ is nonincreasing and concave.

Following the preceding discussions, we can now construct the interactive algorithm in order to derive the satisfying solution for the decision maker from the M-Pareto optimal solution set. The steps marked with an asterisk involve interaction with the decision maker.

Interactive fuzzy satisfying method for fractile criterion model

Step 1. Ask the decision maker to specify the probability thresholds $\alpha_l \in (1/2, 1)$, $l = 1,\ldots,k$ and satisfying levels $\beta_i, i = 1,\ldots,m$ for the chance constrained condition in Eq. (1).

Step 2. After calculating the individual minimum $f_{l,min}$ of $f_l(x)$, $l = 1,\ldots,k$ in Eq. (50), ask the decision maker to determine membership functions $\mu_l(f_l(x))$ for objective functions in Eq. (50), which are nonincreasing and concave.

Step 3. Ask the decision maker to set the initial reference membership levels. If it is difficult for the decision maker to specify them appropriately, set $\bar{\mu}_l = 1,\ l = 1,\dots,k$.

Step 4. Solve the augmented minimax problem Eq. (54) corresponding to the reference membership levels $\bar{\mu}_l = 1,\ l = 1,\dots,k$.

Step 5. The decision maker is supplied with the corresponding M-Pareto optimal solution. If the decision maker is satisfied with the current membership function values of the M-Pareto optimal solution, stop. Otherwise, ask the decision maker to update the reference membership levels $\bar{\mu}_l = 1, l = 1,\dots,k$ by considering the current membership function values $\mu_l(f_l(\mathbf{x}^*))$, and return to Step 4.

Here it should be stressed to the decision maker that any improvement of one membership function can be achieved only at the expense of at least one of the other membership functions.

7. NUMERICAL EXAMPLE

In this section, being limited by space, we only present an illustrative numerical example of an interactive fuzzy satisfying method using the

$$\left.\begin{array}{ll}
\text{Minimize} & (c_1^1 + t_1(\omega)c_1^2)x + (\alpha_1^1 + t_1(\omega)\alpha_1^2) \\
\text{Minimize} & (c_2^1 + t_2(\omega)c_2^2)x + (\alpha_2^1 + t_2(\omega)\alpha_2^2) \\
\text{Minimize} & (c_3^1 + t_3(\omega)c_3^2)x + (\alpha_3^1 + t_3(\omega)\alpha_3^2) \\
\text{subject to} & a_1 x \le b_1(\omega) \\
 & a_2 x \le b_2(\omega) \\
 & a_3 x \le b_3(\omega) \\
 & a_4 x \le b_4(\omega) \\
 & a_5 x \le b_5(\omega) \\
 & a_6 x \le b_6(\omega) \\
 & a_7 x \le b_7(\omega) \\
 & x \ge 0
\end{array}\right\} \qquad (55)$$

probability maximization model. Concerning numerical examples for other models, the interested readers might refer to the corresponding papers (Sakawa et al., 2001;2000; 2002; 2004; Sakawa et al., 2003b; Sakawa and Kato, 2002). Consider the following multi-objective linear programming problem involving random variable coefficients (3 objectives, 10 variables, and 7 constraints).

In this problem, $t_1(\omega)$, $t_2(\omega)$, and $t_3(\omega)$ are Gaussian random variables $N(4,2^2)$, $N(3,3^2)$, and $N(3,2^2)$, where $N(\alpha,\beta^2)$ stands for a Gaussian random variable having mean α and variance β^2.

The right-hand side $b_i(\omega)$, $i=1,\ldots,7$ are also Gaussian random variables $N(164,30^2)$, $N(-190,20^2)$, $N(-184,15^2)$, $N(99,22^2)$, $N(150,17^2)$, $N(154,35^2)$ and $N(142,42^2)$. On the other hand, constant coefficients in (55) are shown in Table 1 and Table 2.

Table 1. Constant Coefficients of Objective Function in Eq. (55)

c_1^1	19	48	21	10	18	35	46	11	24	33	α_1^1	−18
c_1^2	3	2	2	1	4	3	1	2	4	2	α_1^2	5
c_2^1	12	−46	−23	−38	−33	−48	12	8	19	20	α_2^1	−27
c_2^2	1	2	4	2	2	1	2	1	2	1	α_2^2	6
c_3^1	−18	−26	−22	−28	−15	−29	−10	−19	−17	−28	α_3^1	−10
c_3^2	2	1	3	2	1	2	3	3	2	1	α_3^2	4

First, according to Step 1, the decision maker determines the satisfying levels $\beta_i, i=1,\ldots,7$ for each of the constraints in Eq. (55). The hypothetical decision maker in this example specifies the satisfying levels as $(\beta_1,\beta_2,\beta_3,\beta_4,\beta_5,\beta_6,\beta_7)^T = (0.85, 0.95, 0.80, 0.90, 0.85, 0.80, 0.90)^T$.

Second, according to Step 2, the individual minimum $\bar{z}_{l,min}$ and maximum $\bar{z}_{l,max}$ of objective functions $\mathrm{E}[z_l(\boldsymbol{x},\omega)]$, $l=1,\ldots,k$, are calculated under the chance constrained conditions corresponding to the satisfying levels. Each value is obtained as

$$\bar{z}_{1,min} = 1819.571,\ \bar{z}_{1,\max} = 4221.883,\ \bar{z}_{2,\min} = 286.617,\ \bar{z}_{2,\max} = 1380.041,$$
$$\bar{z}_{3,\min} = -1087.249,\ \bar{z}_{3,\max} = -919.647.$$

Table 2. Constant Coefficients of Constraints in Eq. (55).

$\boldsymbol{a}_1$	12	−2	4	−7	13	−1	−6	6	11	−8
$\boldsymbol{a}_2$	−2	5	3	16	6	−12	12	4	−7	−10
$\boldsymbol{a}_3$	3	−16	−4	−8	−8	2	−12	−12	4	−3
$\boldsymbol{a}_4$	−11	6	−5	9	−1	8	−4	6	−9	6
$\boldsymbol{a}_5$	−4	7	−6	−5	13	6	−2	−5	14	−6
$\boldsymbol{a}_6$	5	−3	14	−3	−9	−7	4	−4	−5	9
$\boldsymbol{a}_7$	−3	−4	−6	9	6	18	11	−9	−4	7

Third, according to Step 3, the individual minimum $p_{l,min}$ and maximum $p_{l,max}$ of $p_l(\boldsymbol{x}), l=1,\ldots,k$ in the multi-objective probability maximization problem Eq. (27) are calculated as $p_{1,\min}=0.002$, $p_{1,\max}=0.880$, $p_{2,\min}=0.328$, $p_{2,\max}=0.783$, $p_{3,\min}=0.002$, and $p_{3,\max}=0.664$.

The decision maker subjectively determines membership functions to quantify fuzzy goals for objective functions. Here, the following linear membership function is adopted:

$$\mu_l(p_l(\mathbf{x}))=\frac{p_l(\mathbf{x})-p_{l,0}}{p_{l,1}-p_{l,0}}$$

In this chapter, parameters $p_{l,1}$, $p_{l,0}$ in linear membership functions $\mu_l(\cdot)$, $l=1,\ldots,k$ are determined as

$$p_{1,1}=p_1(\mathbf{x}_{1,\max})=0.880,\ p_{1,0}=\min_{l=2,3}\{p_1(\mathbf{x}_{l,\max})\}=0.502$$

$$p_{2,1}=p_2(\mathbf{x}_{2,\max})=0.783,\ p_{2,0}=\min_{l=1,3}\{p_2(\mathbf{x}_{l,\max})\}=0.060$$

$$p_{3,1}=p_3(\mathbf{x}_{3,\max})=0.664,\ p_{3,0}=\min_{l=1,2}\{p_3(\mathbf{x}_{l,\max})\}=0.446$$

by using Zimmermann's method (Zimmermann, 1978).

According to Step 4, the decision maker specifies the initial reference membership levels $(\bar{\mu}_1,\bar{\mu}_2,\bar{\mu}_3)$ as $(1.00, 1.00, 1.00)$.

Next, according to Step 5, in order to find the optimal solution $\boldsymbol{x}^*$ to the minimax problem (24) for $(\bar{\mu}_1,\bar{\mu}_2,\bar{\mu}_3)=(1.00,\ 1.00,\ 1.00)$, after v^* is calculated by solving the problem (28) using the bisection method and phase one of the two-phase simplex method, the linear programming problem (31) is solved by the simplex method. The obtained solution is shown at the second column in Table 3.

Table 3. Process of Interaction

Interaction	1st	2nd	3rd
$\bar{\mu}_1$	1.000	1.000	0.950
$\bar{\mu}_2$	1.000	1.000	1.000
$\bar{\mu}_3$	1.000	0.900	0.900
x_1	15.590	15.665	15.789
x_2	2.120	2.328	2.389
x_3	0.000	0.000	0.000
x_4	0.254	0.042	0.071
x_5	0.000	0.000	0.000
x_6	6.247	6.282	6.388
x_7	0.207	0.142	0.155
x_8	14.176	14.079	13.998
x_9	1.612	1.301	1.236
x_{10}	17.932	17.733	17.694
$\mu_1(p_1(\boldsymbol{x}))$	0.5747	0.6177	0.5948
$\mu_2(p_2(\boldsymbol{x}))$	0.5732	0.6172	0.6436
$\mu_3(p_3(\boldsymbol{x}))$	0.5733	0.5170	0.5435
$p_1(\boldsymbol{x})$	0.719	0.736	0.727
$p_2(\boldsymbol{x})$	0.474	0.506	0.525
$p_3(\boldsymbol{x})$	0.571	0.559	0.565
$-\partial\mu_1 / \partial\mu_2$	0.060	0.060	0.060
$-\partial\mu_1 / \partial\mu_3$	0.831	0.801	0.816

According to Step 6, the hypothetical decision maker cannot be satisfied with this solution, particular, he wants to improve $\mu_1(\cdot)$, $\mu_2(\cdot)$ at the sacrifice of $\mu_3(\cdot)$. Thus, the decision maker updates the reference membership levels to (1.00, 1.00, 0.90) and returns to Step 5. The result for the updated reference membership levels is shown at the third column in Table 3.

The decision maker is still discontented with the value of $\mu_2(p_2(x))$. Since the sensitivity of $\mu_1(p_1(x))$ to $\mu_2(p_2(x))$ is higher than that of $\mu_3(p_3(x))$ from the trade-off information $-\partial\mu_1 / \partial\mu_2 = 0.060$ and $-\partial\mu_1 / \partial\mu_3 = 0.801$, he updates the reference membership levels to

(0.95, 1.00, 0.90) to improve $\mu_2(p_2(x))$ at the expense of $\mu_1(p_1(x))$. By repetition of such interaction with the decision maker, in this example, a satisfying solution is obtained at the third interaction.

8. SOME EXTENSIONS

So far, we have discussed interactive fuzzy satisfying methods for multi-objective stochastic linear programming problems by making use of several stochastic models in chance constrained programming. As an alternative approach, the authors have proposed an interactive fuzzy satisfying method through a simple recourse model (Sakawa et al., 2001). Extensions of the proposed methods to more general cases such as multi-objective stochastic integer programming problems can be found in our papers (Kato et al., 2004a; 2004b; Perkgoz et al., 2003; 2004). For more extensions to two-level stochastic linear programming problems, the readers might refer to our papers (Kato et al., 2004c; Sakawa et al., 2003a; Wang et al., 2004).

9. CONCLUSION

In this chapter, we focused on multi-objective linear programming problems involving random variable coefficients. For transforming the original stochastic programming into deterministic ones, several stochastic models such as an expectation-optimization model, a variance minimization model, a probability maximization model, and a fractile criterion optimization model for chance constrained conditions are introduced.

As a fusion of stochastic approaches and fuzzy ones, assuming that the decision maker has fuzzy goals for each of the objective functions in the transformed problems, several interactive fuzzy satisfying methods for deriving a satisfying solution for the decision maker from the Pareto optimal solution set are presented. Through illustrative numerical examples, the feasibility of the proposed methods are demonstrated

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AN INTERACTIVE ALGORITHM FOR DECOMPOSING: THE PARAMETRIC SPACE IN FUZZY MULTI-OBJECTIVE DYNAMIC PROGRAMMING PROBLEMS

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Abstract: The aim of this chapter is to study the stability of multi-objective dynamic programming (MODP) problems with fuzzy parameters in the objective functions and in the constraints. These fuzzy parameters are characterized by fuzzy numbers. For such problems, the concept and notion of the stability set of the first kind in parametric nonlinear programming problems are redefined and analyzed qualitatively under the concept of α-Pareto optimality. An interactive fuzzy decision-making algorithm for the determination of any subset of the parametric space that has the same corresponding α-Pareto optimal solution is proposed. A numerical example is given to illustrate the method developed in the chapter.

Key words: Fuzzy sets, Monte Carlo simulation, grey-related analysis, data mining

1. INTRODUCTION

Most practical vector optimization problems contain measured or estimated values that are represented by the different coefficients of the objectives and constraints. Such values may not be accurate enough to the errors in measuring, or estimating these values can lead to a false solution or a solution far from the exact solution of the considered problem. So, if

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after solving the problem an error is discovered or some factors are changed that affect these coefficients, the problem has to be solved again.

Stability analysis covers this difficulty. It tells us what coefficients affect the solution greatly if they are changed and what coefficients have negligible effects on the solution.

In this chapter, we study the stability of multiobjective dynamic programming (MODP) problems with fuzzy parameters in the objective functions and in the constraints. These fuzzy parameters are characterized by fuzzy numbers. For such problems, concept and notion of the stability set of the first kind in parametric nonlinear programming problems are redefined and analyzed qualitatively under the concept of α-Pareto optimality. An interactive fuzzy decision making algorithm for the determination of any subset of the parametric space which has the same corresponding α-Pareto optimal solution is proposed. A numerical example is given to illustrate the method presented.

2. PROBLEM FORMULATION

In this chapter, the fuzzy multiobjective dynamic programming (FMODP) problem is considered. Fuzzy vector-minimization problem (FVMP) involving fuzzy parameters in the objective functions and in the constraints (see Abo-Sinna, 1998, 1992, 2004; Bellman, 1957; Bellman and Dreyfus, 1962; Carraway et al., 1990; Chankong, 1981; Cohon, 1978; Deng Feng and Chuntian, 2004; Esogbue, 1983; Henig, 1983; Hussein and Abo Sinna, 1993; 1995; Larson and Casti, 1978; 1982; Mangasarian, 1969; Osman and El-Banna, 1993; Saad, 1995; Su and Hsu, 1991; Tauxe et al., 1979) are selected:

FVMP:

Minimize

$$F_q\left(f_{q1}\left(x_1 , \tilde{a}_1\right) ,\ldots, f_{qN}\left(x_N , \tilde{a}_N\right)\right) , \quad q=1 ,\ldots, Q , \quad Q\geq 2 \tag{1}$$

subject to

$$G_m\left(g_{m1}\left(x_1 , \tilde{b}_1\right) ,\ldots, g_{mN}\left(x_N , \tilde{b}_N\right)\right)\leq 0 , \qquad m=1 ,\ldots, M , \tag{2}$$

$$x_n \in X_n \quad ,n=1 ,\ldots, N,$$

where for each $n = 1,....,N, X$ is a subset of R^{kn} ; x_n is a k_n vector, the objective functions F_q, $q = 1,...,Q$ and the constraint functions G_m, $m = 1,...,M$ are convex real valued functions of the class $c^{(1)}$ on R^N and f_{qn}, g_{mn} , $q = 1,...,Q$, $m = 1,...,M$, $n = 1,...,N$ are real valued functions on X_n, and $a = (\tilde{a}_{11}, \tilde{a}_{22},...,\tilde{a}_{qn})$, $b = (\tilde{b}_{11}, \tilde{b}_{22},...\tilde{b}_{qn})$, $q = 1,...,Q$, $n = 1,...,N$ represent the vectors of fuzzy parameters in the objective functions $f_{qn}(x_n, \tilde{a}_{qn})$ and in the constraint functions $g_{mn}(x_n, \tilde{b}_{qn})$, respectively.

These fuzzy parameters are assumed to be characterized as the fuzzy numbers introduced by Dubois and Prade (1980). It is appropriate to recall here that a real fuzzy number $\tilde{p}$ is a convex continuous fuzzy subset of the real line whose membership function $\mu_{\tilde{p}}(p) = 0$ is defined by (see Dubois and Prade, 1980; Sakawa and Yano, 1990; Zimmerman, 1985; 1987]):

1. A continuous mapping from real set R to the closed interval [0,1],
2. $\mu_{\tilde{p}}(p) = 0$ for all $p \in (-\infty, p_1]$,
3. strictly increasing on $[p_1, p_2]$,
4. $\mu_{\tilde{p}}(p) = 1$ for all $p \in [p_2, p_3]$,
5. strictly decreasing on $[p_3, p_4]$
6. $\mu_{\tilde{p}}(p) = 0$ for all $p \in [p_4, +\infty]$.

A possible shape of fuzzy number $\tilde{p}$ is illustrated in Figure 1. Now, we assume that $\tilde{a}_{qn}$ and $\tilde{b}_{qn}$ in the $FVMP$ are fuzzy numbers whose membership functions are $\mu_{\tilde{a}_{qn}}(a_{qn})$ and $\mu_{\tilde{b}_{qn}}(b_{qn})$ respectively, for simplicity are $\mu_{\tilde{a}}(a)$ and $\mu_{\tilde{b}}(b)$. Here, we assume that the membership function $\mu_{\tilde{p}}(p)$ is differentiable on $[p_1, p_4]$ and the problem ($FVMP$) is stable (see Rockafellar, 1967; Sakawa and Yano, 1990).

Now, we can introduce the definition of α-level set or α-cut of the fuzzy numbers $\tilde{a}_{qn}$ and $\tilde{b}_{qn}$ $(q = 1,...,Q,\ n = 1,...,N)$ (see Dubois and Prade, 1980).

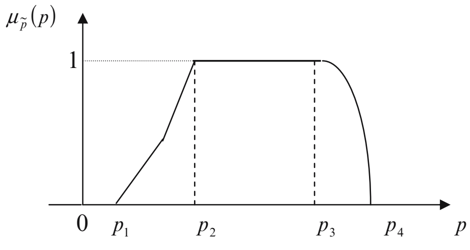

Figure 1. Membership function of fuzzy number

DEFINITION 1. (α-LEVEL SET)

The α-level set of the fuzzy numbers $\tilde{a}_{qn}$ is defined as the ordinary set $L_\alpha(\tilde{a})$ for which the degree of all its component membership functions exceeds the level $\alpha \in [0, 1]$

$$L_\alpha(\tilde{a}) = \left\{ a \ \backslash \ \mu_{\tilde{a}_{qn}}(a_{qn}) \geq \alpha \ , q = 1 \ ,..., \ Q \ , \ n = 1 \ ,..., \ N \right\}$$

Similarly, the α-level set of the fuzzy numbers $\tilde{b}_{qn}$ is defined as the ordinary set $L_\alpha(\tilde{b})$ for which the degree of all its component membership functions exceeds the level $\alpha \in [0,1]$ (see Zimmermann, 1985;1987)

$$L_\alpha(\tilde{b}) = \left\{ b \ \backslash \ \mu_{\tilde{b}_{qn}}(b_{qn}) \geq \alpha \ , q = 1 \ ,..., \ Q \ , \ n = 1 \ ,..., \ N \right\}$$

Similarly, the α-level set of the fuzzy numbers $\tilde{a}_{qn}$ and $\tilde{b}_{qn}$ is defined as the ordinary set $L_\alpha(\tilde{a}, \tilde{b})$ for which the degree of all its component membership functions exceeds the level $\alpha \in [0, 1]$

$$L_\alpha(\tilde{a} \ , \ \tilde{b}) = \left\{ \begin{array}{l} (a \ , \ b) \backslash \mu_{\tilde{a}_{qn}}(a_{qn}) \geq \alpha, \ \mu_{\tilde{b}_{qn}}(b_{qn}) \geq \alpha, \quad q = 1 \ ,..., \ Q \ , \\ n = 1,..., \ N \end{array} \right\}.$$

Obviously, we have the following property for the level set: $\alpha_1 \leq \alpha_2$ if and only if $L_{\alpha_1}(\tilde{a}, \tilde{b}) \supseteq L_{\alpha_2}(\tilde{a}, \tilde{b})$.

As can be seen from the Definition 1, the α-level set $L_\alpha(\tilde{a}, \tilde{b})$ is the set of the closed intervals depending on the level $\alpha \in [0 \ , \ 1]$.

For a certain degree of $\alpha \in [0,1]$, the problem ($FVMP$) can be written in the following nonfuzzy parametric multiobjective dynamic programming problems (see Dauer and Osman, 1985; Osman and Dauer, 1983) depending on the parameters $(a, b) \in L_\alpha(\tilde{a}, \tilde{b})$, as was done by Sakawa and Yano (1990):

$(\alpha - VMP)$:

$$\text{Minimize} \quad F_q\left(f_{q1}(x_1, a_1) \ ,..., \ f_{qN}(x_N, a_N)\right), \quad q = 1 \ ,..., \ Q, Q \geq 2 \tag{3}$$

subject to

$$G_m\left(g_{m1}(x_1,b_1),\ldots,g_{mN}(x_N,b_N)\right)\le 0, \quad m=1\ldots,M \tag{4}$$

$$x_n \in X_n, n=1,\ldots,N, \ (a,b)\in L_\alpha\left(\tilde{a},\tilde{b}\right)$$

Since (FVMP) is stable, the problem (α-VMP) is stable.

It should be emphasized here that in the problem (α-VMP), the parameters a and b are treated as decision variables rather than as constants.

Separability and monotonicity of functions have been used for deriving the recursive formula of dynamic programming (see Abo-Sinna and Hussein 1994; 1995). Definition of these properties for the problem (α-VMP) is given below.

DEFINITION 2. (SEPARABILITY AND MONOTONICITY)

The objective function F_q is said to be separable if there exist functions $F_q^n, n=1,\ldots,N$, defined on R^n and functions Q_q^n, $n=2,\ldots,N$, defined on R^2 satisfying, for $n=2,\ldots,N$,

$$F_q^n\left(f_{q1}(x_1,a_1),\ldots,f_{qn}(x_n,a_n)\right)=$$

$$Q_q^n\left[F_q^{n-1}\left(f_{q1}(x_1,a_1),\ldots,f_{qn-1}(x_{n-1},a_{n-1})\right),f_{qn}(x_n,a_n)\right] \tag{5}$$

and

$$F_q^N\left(f_{q1}(x_1,a_1),\ldots,f_{qN}(x_N,a_N)\right)=F_q\left(f_{q1}(x_1,a_1),\ldots,f_{qN}(x_N,a_N)\right).$$

Similarly, the constraint function G_m is separable, if there exist functions G_m^n, $n=1,\ldots,N$, defined on R^n and functions ψ_m^n, $n=2,\ldots,N$, defined on R^2 satisfying, for $n=2,\ldots,N$

$$G_m^n\left(g_{m1}(x_1,b_1),\ldots,g_{mn}(x_n,b_n)\right)$$

$$=\psi_m^n\left[G_m^{n-1}\left(g_{m1}(x_1,b_1),\ldots,g_{mn-1}(x_{n-1},b_{n-1})\right),g_{mn}(x_n,b_n)\right]$$

and

$$G_m^N\left(g_{m1}(x_1,b_1),\ldots,g_{mN}(x_N,b_N)\right)=G_m\left(g_{m1}(x_1,b_1),\ldots,g_{mN}(x_N,b_N)\right)$$

If all objective and constraint functions are separable, we say that the problem $(\alpha\text{-}VMP)$ is separable. Moreover, the functions ϕ_q^n and ψ_q^n are called the separating functions of F and G.

Furthermore, the separation of the problem $(\alpha\text{-}VMP)$ is said to be monotone if all functions ϕ_q^n and ψ_q^n are strictly increasing with respect to the first argument for each fixed second argument. Specifically, for each $y \in R$

$$\phi_q^n(s, y) > \phi_q^n(s', y) \text{ iff } s > s'$$

and

$$\psi_m^n(s, y) > \psi_m^n(s', y) \text{ iff } s > s'$$

For every $q = 1, \ldots, Q$ $m = 1, \ldots, M$ and $n = 2, \ldots, N$.

Based on the definition of α-level set of the fuzzy numbers (see Kacprzyk and Orlovski, 1987; Orlovski, 1984; Zadeh, 1963) the concept of α-Pareto optimal solution to the problem $(\alpha - VMP)$ is introduced in the following definition (see Sakawa and Yano, 1990).

DEFINITION 3. (α-PARETO OPTIMAL SOLUTION)

A point $x^0 = (x_1^0, \ldots, x_N^0)$ is said to be an α-Pareto optimal solution to the problem $(\alpha - VMP)$, if and only if there does not exist another $x = (x_1, \ldots, x_N)$, $(a, b) \in L_\alpha(\tilde{a}, \tilde{b})$ such that

$$F_q\left(f_{q1}(x_1, a_1), \ldots, f_{qN}(x_N, a_N)\right) \le F_q\left(f_{q1}(x_1^0, a_1^0), \ldots, f_{qN}(x_N^0, a_N^0)\right)$$

For all q and

$$F_r\left(f_{r1}(x_1, a_1), \ldots, f_{rN}(x_N, a_N)\right) < F_r\left(f_{r1}(x_1^0, a_1^0), \ldots, f_{rN}(x_N^0, a_N^0)\right)$$

For at least one index $r \in \{1, 2, \ldots, Q\}$, where the corresponding values of parameters $(a^0, b^0) \in L_\alpha(\tilde{a}, \tilde{b})$ are called α-level optimal parameters.

Assumption 1. The problem $(\alpha - VMP)$ is separable and the separation is monotone.

Assumption 2. For every n, $X_n \cap L_\alpha(\tilde{a}, \tilde{b})$ is compact and $F_q^n(f_{q1}(x_1, a_1), \ldots, f_{qn}(x_n, a_n))$, $q = 1, \ldots, Q$ is continuous functions of $(x_1, \ldots, x_n)$, $(a_1, \ldots, a_n)$, and $G_m^n(g_{m1}(x_1, b_1) \ldots, g_{mn}(x_n, b_n))$, $m = 1, \ldots, M$ is continuous functions of $(x_1, \ldots, x_n)$ and $(b_1, \ldots, b_n)$.

The problem $(\alpha - VMP)$ will be treated using one of the existing parametric approaches, i.e., by considering the following nonlinear program with scalar objective (see Chankong and Haimes, 1983) $(\alpha - VMP_\lambda)$:

Minimize

$$\sum_{q=1}^{Q} \lambda_q F_q\left(f_{q1}(x_1, a_1), \ldots, f_{qN}(x_N, a_N)\right) \tag{6}$$

subject to

$$G_m\left(g_{m1}(x_1, b_1), \ldots, g_{mN}(x_N, b_N)\right) \le 0, \quad m = 1, \ldots, M$$

$$(x_1 \in X_1), \ldots, (x_n \in X_n), \quad n = 1, \ldots, N, \ (a, b) \in L_\alpha\left(\tilde{a}, \tilde{b}\right),$$

for some $\lambda \in \Lambda = \left\{\lambda \in R^Q \setminus \sum_{q=1}^{Q} \lambda_q = 1, \lambda_q \ge 0\right\}$.

It is easy to see that the stability of the problem $(\alpha - VMP)$ implies the stability of the problem $(\alpha - VMP_\lambda)$. It is well known that (x^0) is an α-Pareto optimal solution of the problem $(\alpha - VMP)$ with the corresponding α-level optimal parameters $(a^0, b^0) \in L_\alpha\left(\tilde{a}, \tilde{b}\right)$ if there exists $\lambda^0 \ge 0$, $\lambda^0 \ne 0$ such that x^0 is the unique optimal solution of $(\alpha - VMP_{\lambda^0})$ if there exists $\lambda^0 = \left(\lambda_1^0, \ldots, \lambda_Q^0\right) > 0$, provided every $X_n \cap L_\alpha\left(\tilde{a}, \tilde{b}\right)$ is closed and convex .

Let us suppose that every F_q is additive, i.e., for $q = 1, \ldots, Q$, (see Abo-Sinna, 1998)

$$F_q\left(f_{q1}(x_1, a_1), \ldots, f_{qN}(x_N, a_N)\right) = \bar{f}_{q1}(x_1, a_1) + \ldots + \bar{f}_{QN}(x_N, a_N).$$

Then the objective function in the problem $(\alpha - VMP_\lambda)$ becomes

$$\sum_{n=1}^{N} \sum_{q=1}^{Q} \lambda_q \, \bar{f}_{qn}(x_n, a_n) = \sum_{n=1}^{N} \lambda \, f_n(x_n, a_n) \tag{7}$$

If we define real−valued functions $B_n(\lambda,z)$ for each $n=1,...,\ N$, each $\lambda=(\lambda_1,...,\ \lambda_Q)>0$ and $z=(z_1,...z_M)$ by

$$B_n(\lambda\ ,z)=\min\left\{\sum_{i=1}^{n}\lambda\ \ f_i(x_i\ ,\ a_i)\backslash G_m^n(g_{m1}(x_1\ ,\ b_1)\ ,...,\ g_{mn}(x_n\ ,\ b_n))\le z_m\ ,\right.$$

$$\left. m=1\ ,...,\ M,\ x_1\in X_1\ ,...,\ x_n\in X_n\ ,\ \ (a,b)\in L_\alpha(\tilde{a},\tilde{b})\ \right\}$$

Now, we can obtain the recursive relations for $n=2,...,N$:

$$B_n(\lambda,z)=\min_{x_n\in X_n,(a,b)\in L_\alpha(\tilde{a},\tilde{b})}\{B_{n-1}(\lambda,z^{n-1}(x_n,z))+\lambda f_n(x_n,a_n)\}$$

$$\text{where}\quad z^{n-1}(x_n,z)=(z_1^{n-1}(x_n,z),...,z_M^{n-1}(x_n,z)).$$

Assuming monotonicity of G_m^n, let z_m^{n-1} be defined by

$$z_m^{n-1}(x_n\ ,\ z)=\sup\ \left\{\ \xi\in R\ ;\ G_m^n(\xi\ ,g_{mn}(x_n\ ,\ b_n))\le z_m\ ,\ (b_n\ ,...,b_n)\in L_\alpha(\tilde{b})\ \right\}$$
$$m=1\ ,...,\ M.$$

THEOREM 1.

Suppose that Assumption 1 and Assumption 2 hold. Let $(x_1^0,...,x_n^0)$ be any α-Pareto optimal solution of problem $B_n(\lambda^0,z)$ for some $\lambda^0\in\Lambda$, where the corresponding α -level optimal parameters $(a_1^0,...,a_n^0,b_1^0,...,b_n^0)\in L_\alpha((\tilde{a}_1,...,\tilde{a}_n,\tilde{b}_1,...,\tilde{b}_n))$. Then $(x_1^0,...,x_{n-1}^0)$ is an α-Pareto optimal solution of problem $B_{n-1}(\lambda^0,z^{n-1}(x_n,z))$, where the corresponding α -level optimal parameters $(a_1^0,...,a_{n-1}^0,b_1^0,...,b_{n-1}^0)\in L_\alpha((\tilde{a}_1,...,\tilde{a}_{n-1},\tilde{b}_1,...,\tilde{b}_{n-1}))$.

The proof of this theorem is much like that of Theorem 1 in (Mine and Fukushima, 1979).

Using the recursive relations (2) for various values of λ we may find a set of α-Pareto optimal solution of the problem $(\alpha-VMP)$ by obtaining $B_n(\lambda^0,0)$.

2.1 The Stability Set of the First Kind

DEFINITION 4.

Suppose that a certain $\lambda^0 \in \Lambda$ with a corresponding α-Pareto optimal solution (x^0) of the problem $(\alpha - VMP)$, where $(a^0, b^0) \in L_\alpha(\tilde{a}, \tilde{b})$ are the corresponding α-level optimal parameters. Then the stability set of the first kind of the problem $(\alpha - VMP)$ corresponding to (x^0), which is denoted by $S(x^0, a^0, b^0)$, is defined by:

$S(x^0, a^0, b^0) = \{\lambda \in \Lambda \setminus x^0$ is an α-Pareto optimal solution of the problem $(\alpha - VMP)$ with the corresponding α-level optimal parameters $(a^0, b^0)\}$.

THEOREM 2.

If the functions F and G are convex, and $\mu_{\tilde{a}}(a), \mu_{\tilde{b}}(b)$ are concave functions, then the set $S(x^0, a^0, b^0)$, which is the stability set of the first kind of the problem $(\alpha - VMP)$ corresponding to the α-Pareto optimal solution x^0 with the α-level optimal parameters $(a^0, b^0) \in L_\alpha(\tilde{a}, \tilde{b})$, is convex and $S(x^0, a^0, b^0) \cup \{0\}$ is closed. Furthermore, if $S(x^*, a^*, b^*)$ is the stability set of the first kind of the problem $(\alpha - VMP)$ corresponding to the α-Pareto optimal solution x^* with the α-level optimal parameters $(a^*, b^*) \in L_\alpha(\tilde{a}, \tilde{b})$ and $int[S(x^0, a^0, b^0) \cap (S(x^*, a^*, b^*))] \neq \phi$, then $S(x^0, a^0, b^0) = S(x^*, a^*, b^*)$.

The proof of this theorem is similar to the one in Osman (see Caplin and Kornbluth, 1957).

REMARK 1. (Osman, 1977)

It must be noted that the above properties of the stability set of the first kind still hold if the continuity and differentiability assumptions that are imposed on F and G are relaxed.

2.2 Determination of the Stability Set of the First Kind

Let $\lambda^0 \in \Lambda$ with an α-Pareto optimal solution (x^0) of the problem $(\alpha - VMP)$ with the corresponding α-level optimal parameters $(a^0, b^0) \in L_\alpha(\tilde{a}, \tilde{b})$, then according to the Kuhn-Tucker necessary optimality conditions (see Mangarasian, 1969), for the problem $(\alpha - VMP_\lambda)$, it follows that there exists $\lambda \in \Lambda$, $\lambda \geq 0$, $U \in R^M$, $U \geq 0$, $V \in R^q$, $w \in R^q$, $V \geq 0$, $w \geq 0$, such that

$$\lambda^T \frac{\partial F}{\partial x}(x^0, a^0) + U^T \frac{\partial G}{\partial x}(x^0, b^0) = 0 \tag{8}$$

$$\lambda^T \frac{\partial F}{\partial a}(x^0, a^0) - V^T \frac{\partial \mu_a}{\partial a}(a^0) = 0 \tag{9}$$

$$U^T \frac{\partial G}{\partial b}\left(x^0, b^0\right) - W^T \frac{\partial \mu_b}{\partial b}\left(b^0\right) = 0 \tag{10}$$

$$G\left(x^0, b^0\right) \le 0,\ \alpha - \mu_{\tilde{a}}\left(a^0\right) \le 0,\quad \alpha - \mu_{\tilde{b}}\left(b^0\right) \le 0,\ U^T G\left(x^0, b^0\right) = 0$$

$$V^T\left[\alpha - \mu_{\ddot{a}}\left(a^0\right)\right] = 0,\ W^T\left[\alpha - \mu_{\tilde{b}}\left(b^0\right)\right] = 0$$

where η^T stands for the transpose of the vector η. Denote the following sets:

$$A\left(x^0, b^0\right) = \left\{m \setminus G_m\left(g_{m1}\left(x_1^0, b_1^0\right), \dots, g_{mN}\left(x_N^0, b_N^0\right)\right) = 0, \quad m = 1, \dots, M\right\}$$

$J\left(a^0\right) = \left\{(q, n) \setminus \mu_{\tilde{a}}\left(a_q^0\right) = \alpha\right\}$ and

$J\left(b^0\right) = \left\{(q, n) \setminus \mu_{\tilde{b}}\left(b_q^0\right) = \alpha\right\}$.

Then we have the following three linear independent systems of equations:

$$\lambda^T \frac{\partial f}{\partial x}\left(x^0, a^0\right) + \sum_{m \in A\left(x^0, b^0\right)} \mu_m \frac{\partial G_m}{\partial x}\left(x^0, b^0\right) = 0 \tag{11}$$

$$\sum_{q=1}^{Q} \lambda_q \frac{\partial F_q}{\partial a_q}\left(x^0, a^0\right) - V_q \frac{\partial \mu_{a_q}}{\partial a_q}\left(a_q^0\right) = 0 \tag{12}$$

$$U^T \frac{\partial G_m}{\partial b_q}\left(x^0, b^0\right) - W_q \frac{\partial \mu_{b_q}}{\partial b_q}\left(b_q\right) = 0 \tag{13}$$

$$J\left(a^0\right),\ V_q = 0,\ q \notin J\left(a^0\right);\ w_q \ge 0,\ q \in J\left(b^0\right),\ w_q = 0,\ q \notin J\left(b^0\right)$$

System (11) represents the first group of the Kuhn−Tucker conditions and it can be rewritten in the following matrix form :

$$\begin{bmatrix} C' & D' \end{bmatrix} \begin{bmatrix} \lambda \\ \mu \end{bmatrix} = 0 \tag{14}$$

where $C' = [c'_{ij}]$ is an $s \times Q$ matrix, $D' = [d'_{ij}]$ is an $h \times k$ matrix, $\lambda \in R^Q, \mu \in R^K$ $\lambda \geq 0, \lambda \neq 0$ and $\mu \geq 0, U \in R^s, V \in R^h$, where s, h are the cardinalities of $A(x^0, b^0)$ and J, respectively .

Suppose $d'_{ij} = 0, \quad j = 1,..., K, \quad i \in I \subset \{1, 2,..., s\}$, where the cardinal number of I is assumed to be equal to $s - l$. Then we ignore for moment these rows and consider the remaining system which will have the form

$$\begin{bmatrix} C & D \end{bmatrix} \begin{bmatrix} \lambda \\ \mu \end{bmatrix} = 0 \tag{15}$$

Here C and D are matrices of order $l \times Q$ and $l \times k$, respectively. Therefore system (11) together with the condition $\sum_{j=1}^{Q} C'_{ij} \quad \lambda_j = 0, \quad i \in I$ gives system (14), which is equivalent to system (11); hence we give the following two propositions (see Zeleny , 1973; 1982):

PROPOSITION 1.

If $K \geq l$, then

$$S(x^0, a^0, b^0) = \left\{ \lambda \in \Lambda \setminus \begin{pmatrix} (\lambda^T C^T (D_l^T)^{-1})_j \leq 0 \\ j = 1,...,l, \sum_{j=1}^{Q} C'_{ij} \lambda_j = 0, i \in I \end{pmatrix} \right\} \tag{16}$$

where $D = [D_l \, D_2], D_l$ and D_2 are respectively $l \times l$ and $l \times k - l$ matrices and η_j is the element in the jth column of the row vector η.

PROPOSITION 2.

If $K < l$, then

$$S\left(x^0\ ,\ a^0,b^0\right)=\left\{\begin{array}{l} \lambda\in\Lambda\setminus\left(\lambda^T\left[C_2^T-C_1^T(D_1^T)^{-1}\ \ D_2^T\right]\right)_j=0 \\ j=1\ ,\ldots,\ k-l\ ,\left(\lambda^T C_1^T\left(D_1^T\right)^{-1}\right)_j\leq\ 0 \\ j=1,\ \ldots,\ k,\ \sum_{j=1}^{Q}C'_{i\,j}\ \ \lambda_j=0,\quad i\in I \end{array}\right\} \tag{17}$$

REMARK 2.

If λ is normalized by the condition $\sum_{q=1}^{Q}\lambda_q=1,$ then we can add this condition to the set $S\left(x^0,a^0,b^0\right)$ in any one of its form.

2.3 An Algorithm

Now, we can construct an algorithm to determine the stability set $S\left(x^0,a^0,b^0\right)$ of the problem $(\alpha-VMP)$ as follows.

Step 1. Elicit a membership function from the decision maker for each of the fuzzy numbers $\tilde{a}$ and $\tilde{b}$ in the problem $(FVMP)$.

Step 2. Ask the decision maker to select the initial values of $\alpha(0\leq\alpha\leq 1)$.

Step 3. Construct the parametric multi-objective dynamic program general, as the vector minimization problem $(\alpha-VMP)$.

Step 4. Ask the decision maker to choose certain $\lambda^0\in\Lambda$ and by using the recursive relations (2), the decision maker approach can be used to obtain an α-Pareto optimal solution x^0 of the problem $(\alpha-VMP_\lambda)$ by obtaining $B_n(\lambda,0)$, $n=1,\ldots,N$. Suppose that $\left(a^0,b^0\right)\in L_\alpha\left(\tilde{a},\tilde{b}\right)$ is the corresponding α-level optimal parameters (using any available nonlinear programming package, for example, GINO at each stage).

Step 5. Substitute with $\left(x^0,a^0,b^0\right)$ in the Kuhn−Tucker necessary conditions, we obtain system (11), and system (15) can be easily found. Also system (12) can be solved by Gauss-elimination.

Step 6. According to the values of the Lagrange multipliers, we get a) if $s=m+k-l,$ then $S\left(x^0,a^0,b^0\right)=\left\{t\lambda^0\setminus t>0\right\}$; b) if $k\geq l$, then $S\left(x^0,a^0,b^0\right)$ is given by (16), andc) if $k<l$, then $S\left(x^0,a^0,b^0\right)$ is given by Eq. (17)

Step 7. If the DM is satisfied with current solutions, stop. Otherwise, ask the decision maker to update the degree $\alpha = (\alpha + \Delta) \in [0, 1]$ and return to Step 3.

2.4 Numerical Example

Let us consider the following multi-objective dynamic programming problem with fuzzy parameters in the objective functions (in fact, this problem has three stages and three objectives), namely, the fuzzy vector minimization problem is written as follows $(FVMP)$:

$$\text{Minimize } f_1(x, \tilde{a}_1) = (x_1 - \tilde{a}_{11})^2 + x_2^2 + x_3^2$$
$$\text{Minimize } f_2(x, \tilde{a}_2) = (x_1 - 1)^2 + (x_2 + \tilde{a}_{22})^2 + (x_3 - 2)^2$$
$$\text{Minimize } f_3(x, \tilde{a}_3) = 2x_1 + x_2^2 + (x_3 - \tilde{a}_{33}).$$

subject to

$$M_1 = \{x \in R^3 \setminus x_1 + x_2 + x_3 \le 3, \quad x_j \ge 0, \quad j = 1, \ 2, \ 3\}$$

Let the fuzzy parameters be characterized by the following fuzzy numbers:

$$\tilde{a}_{11} = (0,1,3,5), \quad \tilde{a}_{22} = (0,1,4,6), \quad \tilde{a}_{33} = (3,5,9,10).$$

Assume that membership function for each fuzzy number $\tilde{a}$ in problem $(FVMP)$ is defined by

$$\mu_{\tilde{a}}(a) = \begin{cases} 0, & -\infty < a \le p_1 \\ 1 - ((a - p_2)/\ (p_1 - p_2))^2, & p_1 \le a \le p_2 \\ 1, & p_2 \le a \le p_3 \\ 1 - ((a - p_3)/\ (p_4 - p_3))^2, & p_3 \le a \le p_4 \\ 0, & p_4 \le a < \infty \end{cases}$$

Consider the α – level sets or α – cuts of the fuzzy numbers, which are given by

$\mu_{\tilde{a}11}(a_{11}) \geq 0.36$; then we get $0.2 \leq a_{11} \leq 4.6$

$\mu_{\tilde{a}22}(a_{22}) \geq 0.36$; then we get $0.4 \leq a_{22} \leq 5.6$

$\mu_{\tilde{a}33}(a_{33}) \geq 0.36$; then we get $3.4 \leq a_{11} \leq 9.8$

(See Figure 1).

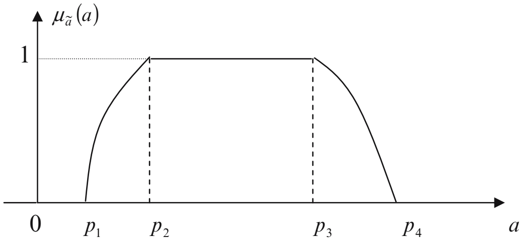

Figure 2. α – cuts of the fuzzy numbers.

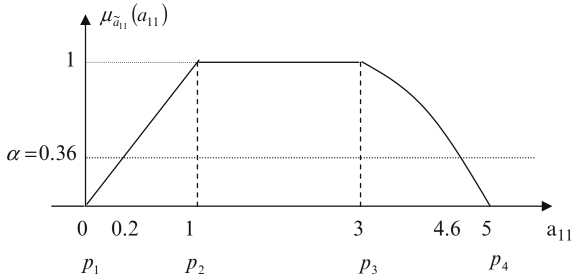

Figure 3. α – cuts of the fuzzy numbers

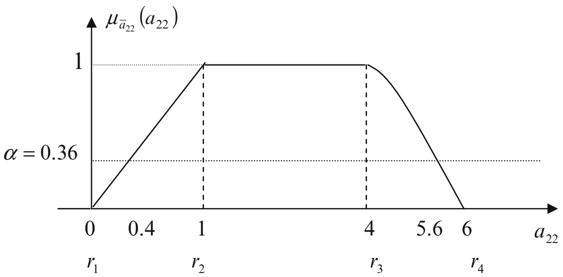

Figure 4. α – cuts of the fuzzy numbers

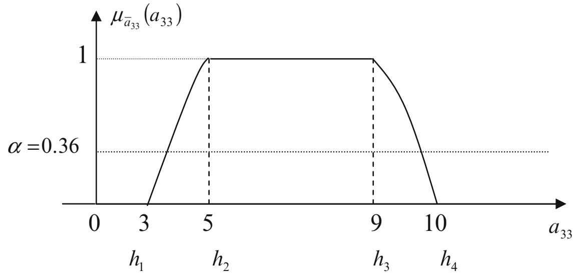

Figure 5. α – cuts of the fuzzy numbers

The nonfuzzy α- vector minimization problem $(\alpha - VMP)$ can be written as follows:

$(\alpha - VMP)$:

Minimize $\{f_1(x,a_1), f_2(x,a_2), f_3(x,a_3)\}$

subject to

$$x_1 + x_2 + x_3 \le 3 \;, \quad 0.2 \le a_{11} \le 4.6 \;, \quad 0.4 \le a_{22} \le 5.6 \qquad (18)$$
$$3.4 \le a_{33} \le 9.8 \;, \quad x_1,\ x_2,\ x_3 \ge 0$$

where

$$f_1(x,a_1) = (x_1 - a_{11})^2 + x_2^2 + x_3^2$$

$$f_2(x,a_2) = (x_1 - 1)^2 + (x_2 + a_{22})^2 + (x_3 - 2)^2$$

$$f_3(x,a_3) = 2x_1 + x_2^2 + (x_3 - a_{33})^2$$

It is easy to see that $(\alpha - VMP)$ satisfies Assumptions 1 and 2. Therefore a dynamic programming approach can be applied for characterizing the α-Pareto optimal solution of the problem $(\alpha - VMP)$.

Using the weighting method (Chankong and Hamines, 1983) then the problem $(\alpha - VMP)$ becomes

$(\alpha - VMP)_\lambda$:

Minimize $\sum_{q=1}^{3} \lambda_q f_q(x,a)$.

Subject to the set of constraints Eq. (18)

i) At $\lambda_0 = (1/3, 1/3, 1/3,)$, the dynamic programming approach has the following steps:

Step 1.

$$B_1(\lambda^0, 0) = \text{ Minimize}\left\{\sum_{q=1}^{3} \lambda_q^0 f_{q1}(x_1, a_{11}) / x_1 \le 3,\ 0.2 \le a_{11} \le 4.6,\ x_1 \ge 0\right\}$$

$$= \text{Minimize } \left\{1/3(x_1 - a_{11})^2 + 1/3(x_1 - 1)^2 + 2/3x_1 / x_1 \le 3,\quad 0.2 \le a_{11} \le 4.6,\quad x_1 \ge 0\ \right\}$$

by using GINO package, the α-Pareto optimal solution to $B_1(\lambda^0, 0)$ be $(x_1^0, a_{11}^0) = (0.1, 0.2)$.

Step 2.

$$B_2(\lambda^0,\ 0) = \text{Min } \left\{\sum_{q=1}^{3} \lambda_q^0 f_{q1}(x_1^0,\ a_{11}^0) + f_{q2}(x_2,\ a_{22}) \backslash x_1^0 + x_2 \le 3\ ,\ x_1^0,\ x_2 \ge 0,\ \ 0.4 \le a_{22} \le 5.6\right\}$$

$$= \text{Min } \left\{0.34 + \tfrac{1}{3}x_2^2 + \tfrac{1}{3}(x_2 + a_{22})^2 + \tfrac{1}{3}x_2^2 \quad \backslash \quad 0.1 + x_2 \le 3,\ \ 0.4 \le a_{22} \le 5.6,\ \ x_2 \ge 0\right\}$$

Hence the α-Pareto optimal solution to $B_1(\lambda^0, 0)$ is $(x_1^0, x_2^0, a_{11}^0, a_{22}^0) = (0.1,\ 0.0,\ 0.2, 0.4)$.

Step 3.

$$B_3(\lambda^0,\ 0) = \text{Minimum } \left\{\sum_{q=1}^{3} \lambda_q^0 f_{q1}(x_1^0,\ a_{11}^0) + f_{q2}(x_2^0,\ a_{22}^0) + f_{q3}(x_3,\ a_{33}) \backslash x_1^0 + x_2^0 + x_3 \le 3\right.$$

$$\left. 3.4 \le a_{33} \le 9.8,\ x_1^0, x_2^0, x_3 \ge 0\right\}.$$

Thus

$$B_3(\lambda^0,\ 0) = \text{Minimum } \left\{\ 0.39333 + \tfrac{1}{3}x_3^2 + \tfrac{1}{3}(x_3 - 2)^2 + \tfrac{1}{3}(x_3 - a_{33})^2 \backslash 0.1 + 0.0 + x_3 \le 3,\right.$$

$$3.4 \le a_{33} \le 9.8,\ \ x_3 \ge 0$$

and $(x_1^0, x_2^0, x_3^0, a_{11}^0, a_{22}^0, a_{33}^0) = (0.1, 0.0, 1.8, 0.2, 0.4, 3.4)$ is the α-Pareto optimal solution to $(\alpha - VMP_\lambda)$ and the optimum objective value equals 2.34.

ii) Determining the stability set of the first kind to the problem $(\alpha - VMP)$:

Since all functions $G_m (m = 1,2,3,4)$ in the problem $(FVMP)$ do not appear the fuzzy number $\tilde{b}$, it is easily seen that the set $S(x^0, a^0, b^0)$, which is the stability set of the first kind of the problem $(\alpha - VMP)$

corresponding to the α-Pareto optimal solution $x^0 = (0.1,0,0)$ with the α-Level optimal parameters $a^0 = (0.2,0.4,3.4)$ is $S(x^0, a^0)$. Therefore, in what follows, we will determine the set $S(x^0, a^0)$.

From the Kuhn−Tucker necessary optimality conditions (system (19) and system (12)) we get:

$$\begin{aligned} -0.2\lambda_1 - 1.8\lambda_2 + 2\lambda_3 + \mu_1 &= 0 \\ 0.8\lambda_2 + \mu_2 &= 0 \end{aligned} \tag{19}$$

$$\begin{aligned} 3.6\lambda_1 - 0.4\lambda_2 - 3.2\lambda_3 + \mu_3 &= 0 \\ 0.2\lambda_1 - v_2 &= 0 \\ 0.8\lambda_2 - v_3 &= 0 \\ 3.2\lambda_3 - v_4 &= 0 \end{aligned} \tag{20}$$

System (19) will be the same as system (21) and (22), i.e., $s = l = k = 3$; thus

$$C = \begin{bmatrix} -0.2 & -1.8 & 2 \\ 0 & 0.8 & 0 \\ 3.6 & -0.4 & -3.2 \end{bmatrix}, D = D_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$C^T (D_1^T)^{-1} = \begin{bmatrix} -0.2 & 0 & 3.6 \\ -1.8 & 0.8 & -0.4 \\ 2 & 0 & -3.2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -0.2 & 0 & 3.6 \\ -1.8 & 0.8 & -0.4 \\ 2 & 0 & -3.2 \end{bmatrix}$$

$$\lambda\ C\ (D\ )^{-1} = (\lambda_1,\ \lambda_2,\ \lambda_3) \begin{pmatrix} -0.2 & 0 & 3.6 \\ -1.8 & 0.8 & -0.4 \\ 2 & 0 & -3.2 \end{pmatrix}$$

$$= (-0.2\lambda_1 - 1.8\lambda_2 + 2\lambda_3,\quad 0.8\lambda_2,\quad 3.6\lambda_1 - 0.4\lambda_2 - 3.2\lambda_3)\ .$$

We get $\lambda_1 \geq 9\lambda_2 - 10\lambda_3$, $\lambda_2 \geq 8\lambda_3 - 9\lambda_1$, $\lambda_1 + \lambda_2 + \lambda_3 = 1$. Now we can solve system (20) as follows:

$$v_1 = 0,\ 0.2\lambda_1 - v_2 = 0,\ 0.8\lambda_2 - v_3 = 0,\ 3.2\lambda_3 - v_4 = 0.$$

From

$$\lambda_1 = \lambda_2 = \lambda_3 = \tfrac{1}{3} = 5v_2 = \tfrac{10}{3}v_3 = \tfrac{10}{32}v_4 \text{ we have } v_2 = \tfrac{1}{15},\, v_3 = \tfrac{8}{30} = \tfrac{4}{15}$$

and

$$v_4 = \tfrac{32}{30} = \tfrac{16}{15}\text{. If } c_1 \le a_1^0 \le c_2\,,\, d_1 \le a_2^0 \le d_2\,, e_1 \le a_3^0 \le e_2\,,$$

then

$$V_1(0.2-c_2) = V_2(c_1-0.2) = V_3(0.4-d_2) = V_4(d_1-0.4) = V_5(3.4-e_2) = v_6(e_1-3.4) = 0\ ,$$
i.e. $0.2 \le c_2,\ \ c_1 = 0.2,\ \ 0.4 \le d_2,\ \ d_2 = 0.4,\ \ 3.4 \le e_2,\ \ e_2 = 3.4.$

The stability set of the first kind corresponding to $\left(x^0, a^0\right) = (0.1, 0.0, 1.8, 0.2, 0.4, 3.4)$ takes the form:

$$S(0.1,\ 0.0,\ 1.8,\ 0.2,\ 0.4,\ 3.4) =$$
$$\{(\lambda,\ p,\ r,\ h)\backslash \lambda_1 \ge 9\lambda_2 - 10\lambda_3,\ \ \lambda_2 \ge 8\lambda_3 - 9\lambda_1,\ \ \lambda_1 + \lambda_2 + \lambda_3 = 1$$
$$p_1 < p_2 = 1 - 4p_1 < p_3 < p_4,\ \ 0 < p_1 < 0.2,$$

$$r_1 < r_2 = 1 - 15r_1 < r_3 < r_4,\ \ 0 < r_1 < 0.4$$
$$h_1 < h_2 = 17 - 4h_1 < h_3 < h_4,\ \ 0 < h < 3.4\}$$

It must be observed here that, if the decision maker is not satisfied with the current value of the degree α of the $\alpha-$ Pareto optimal solution, it is possible for the decision maker to continue the same procedure in this manner until the decision maker is satisfied with the current value of the degree α of the $\alpha-$ Pareto optimal solution.

3. CONCLUSIONS

In this chapter, the stability of multi-objective dynamic programming (MODP) problems with fuzzy parameters in the objective functions and in the constraints has been studied. An interactive fuzzy decision-making algorithm for the determination of any subset of the parametric space that has the same corresponding $\alpha-$ Pareto optimal solution has been proposed. Interactive algorithms have a significant potential for the fuzzy research in the future.

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GOAL PROGRAMMING APPROACHES FOR SOLVING FUZZY INTEGER MULTI-CRITERIA DECISION-MAKING PROBLEMS

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Abstract: Multicriteria decision making can be divided into two parts: multi-attribute decision analysis and multi-criteria optimization. When the number of the feasible alternatives is large, we use multi-criteria optimization. On the other hand, multi-attribute decision analysis is most often applicable to problems with a small number of alternatives in an environment of uncertainty. In this chapter, a goal programming approach was analyzed to solve fuzzy integer multi-criteria decision-making problems.

Key words: Goal programming, integer multi-criteria decision-making problem, iterative goal programming approach, fuzzy integer multi-criteria decision making problem

1. INTRODUCTION

The term "multi-criteria decision making" (MCDM) encompasses a wide variety of problems. Multi-criteria decision making is concerned with the methods and procedures by which multi-criteria can be formally incorporated into the analytical process.

Multi-criteria decision making has, however, two distinct halves: one half, is multi-attribute decision analysis, and the other is multi-criteria optimization (multi-objective mathematical programming).

Multi-attribute decision analysis is most often applicable to problems with a small number of alternatives in an environment of uncertainty. Multi-criteria optimization is often applied to deterministic problems in which the number of feasible alternatives is large.

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In recent years research has been carried out in solving multi-criteria integer programming problems, but whereas some has been classified as such, some has appeared in terms such as decision theory.

Treating integer multi-criteria decision-making problems can be classified into three main approaches: vector optimization (multi-objective optimization), goal programming, and interactive approaches. Most of the current research is directed mainly toward the interacting approaches trying to avoid the drawbacks in the other two approaches. Also, the current research includes the stochastic and fuzzy cases.

Most decision problems have multiple objectives that cannot be optimized simultaneously due to the inherent conflict between these objectives. Such problems involve making trade-off decisions to get the "best compromise" solution. Goal programming is a powerful approach that has been proposed for the modeling, solution, and analysis of the multi-criteria decision-making problems. There are a wide variety of goal programming models, including weighted goal programming (Charnes and Cooper, 1961; Ignizio, 1983) lexicographic goal programming, i.e., the use of the so-called "preemptive priority" concept (Ignizio, 1976; 1983), minimax goal programming includes fuzzy programming (Zimmerman, 1978) and interactive goal programming that is used to generate a subset of the nondominated solutions (Ignizio, 1981; Steuer, 1978).

Since goal programming now encompasses any linear, integer, zero-one, or nonlinear multi-objective problem (for which preemptive priorities may be established), the field of applications is wide open. The recent increase in interest in this area has already led to a large number of and a wide variety of actual and proposed applications. For purpose of illustration, we list just a few of these below, and the reader is referred to (Ignizio, 1978):

- Aggregate planning and work force (Dauer and Osman, 1981).
- Qualitative programming for selecting decisions (Zahedi, 1987).
- Curve and response surface fitting (Ignizio, 1977).
- Media planning (Chranes et al., 1968).
- Manpower planning (Chanes and Nilhaus, 1968).
- Program selection (Satterfield and Ignizio, 1974).
- Hospital administration (Lee, 1971).
- Academic resource allocation (Schroeder, 1974).
- Municipal economic planning (Lee and Sevebeck, 1971).
- Transportation problems (Lee and Moore, 1973).
- Energy/water resources (Elchak and Raphael, 1977).
- Radar system design (Ignizio and Satterfield, 1977).
- Sonar system design (Wilson and Ignizio, 1977).

- Planning in wood products (Inoue and Eslick, 1975).
- Portfolio selection (Kumar and Philippatos, 1975)
- Determination of time standards (Mashimo, 1977).
- Development of cost estimating relationship (Ignizio, Inpress).
- Urban renewal planning (Lee and Keown, 1976).
- Merger strategy (Salkin and Jones, 1972).
- Multi-plant/product aggregate production loading (Johnson, 1976).
- BMD systems design (Ignizio and Satterfield, 1977).
- Multi-objective facility location (Harnett and Ignizio, 1972).
- Free flight rockets (Ignizio, 1975a).
- Solar heating and cooling (Ignizio, 1975b).
- Natural gas well sitting (Gochnour, 1976).
- Maintenance level determination (Younis, 1977).
- A Pennsylvania coal model (Kirtland et al., 1977).

All of these applications have one thing in common: they could be forced onto a traditional single-objective model if one so wished. However, those investigating these problems believed that they truly involved multiple, conflicting objectives and were thus most naturally modeled as a goal programming problem (Ignizio, 1978).

2. INTEGER MULTICRITERIA DECISION-MAKING PROBLEM AS A GOAL PROGRAMMING MODEL

The integer multi-criteria decision-making problem (IMCDM) can be formulated mathematically as follows:

(IMCDM):

Maximize $Z(x)$

subject to $x \in X$

where Z: $R^n \to R^k$, $Z(x) = (z_1(x), z_2(x), ..., z_k(x))$ is a vector-valued criterion with $z_i(x)$, $i = 1, 2, ..., k$ which are real-valued objective functions and X is feasible set. This set might be, for example, of the form:

$$X = \{ x \in R^n \mid Ax \le b, x \ge 0 \text{ and integer } \}$$

where A is an $(m \times n)$ matrix of constraint function coefficients; x is an $(n \times 1)$ vector of the integer decision variables; b is an $(m \times 1)$ vector of constraint right-hand sides, whose components specify the available resource; and R^n is the set of all ordered n-tuples of real numbers. It is assumed in problem (IMCDM) that the feasible set X is bounded.

The imperative "maximize" in problem (IMCDM) is understood to mean: Find the set of all solutions that have (roughly) the property that increasing the value of one objective $z_k(x)$ decreases the value of at least one other objective function. This set is usually called an *efficient (or nondominated, noninferior, Pareto-optimal, functional-efficient)* set. This set is a surrogate for an optimal solution to a usual optimization problem with a single objective function. The meaning of an efficient solution is given in the following definition.

DEFINITION 1.

A point $x^* \in X$ is said to be an efficient solution of problem (IMCDM) if there exists no other $x \in X$ such that $Z(x) \geq Z(x^*)$ and $Z(x) \neq Z(x^*)$ (see Chankong and Haims, 1983; Cohon, 1978; Geoffrion, 1968, Hwang and Masud, 1979).

Now, let us express the ith objective function in the form: $z_i(x) = c_i^t x$, where the superscript t denotes the transpose and c_i is an n-vector defined as the vector of the coefficients of the ith objective function.

In goal programming, rather than attempting to optimize the objective criteria directly, the decision maker sets to minimize the deviations between goals and levels of achievement within the given set of constraints. Thus, the objective function becomes the minimization of these deviations on the relative importance assigned to them.

Problem (IMCDM) can be transformed into the following integer linear goal programming model (ILGP) consisting of k goals:

(ILGP):

Find x to achieve:

$z_1(x) = h_1$

$z_2(x) = h_2$

.

.

.

$z_k(x) = h_k$

subject to

$x \in X$

where $h_1, h_2, \ldots, h_k$ are scalars and represent the desired achievement levels of the objective functions that the decision maker wishes to attain provided that $z_{*_i} \prec h_i \prec z^{*i}, \quad i = 1,2,\ldots,k.$

Note that z^* and z_* provide upper and lower bounds on the objective function values and hence are a great source of information for the decision maker. These bounds can easily be determined by solving:

$$\begin{aligned} \text{Maximize} \quad & z_i(x) \\ \text{subject to} \quad & \\ & Ax \le b, \\ & x \ge 0 \text{ and integer.} \end{aligned} \tag{1}$$

The solution of problem (1), (x_i^*, z_i^*) , is known in the literature as the *ideal solution.* Let $z_{ji} = z_i(x_j);$ then

$$z_{*i} = \min_{\{i\}} z_{ji}, \qquad j = 1,2,\ldots,k. \tag{2}$$

DEFINITION 2.

The goals are ranked as follows: if $i \prec j$ then goal i, $c_i^t x = h_i,$ has a higher priority than goal j , $c_j^t x = h_j$, (see preemptive priorities Charnes and Cooper, 1961; Lee, 1972).

3. AN ITERATIVE GOAL PROGRAMMING APPROACH FOR SOLVING (IMCDM) PROBLEM

In order to solve the integer linear goal program (ILGP) by the iteration algorithm developed in Dauer and Krueger (1977) together with the Gomory's fractional cut shown in Klein and Holm (1978, 1979), we first solve the integer linear optimization problem associated with the first goal viz:

P_1:

minimize $\quad L_1 = d_1^- + d_1^+$

subject to

$$c_1^t x + d_1^- - d_1^+ = h_1,$$
$$Ax \le b,$$
$$d_1^- \ge 0,\ d_1^+ \ge 0, x \ge 0, \text{ and integer}$$

where d_1^- and d_1^+ are the underattainment and the overattainment, respectively, of the first goal where $d_1^- d_1^+ = 0$.

Suppose this problem has integer optimal value $L_1^* = d_1^{-*} + d_1^{+*}$ with at least one value d_1^{-*} or d_1^{+*} nonzero.

Now, the attainment problem for goal 2 is equivalent to the integer optimization problem P_2, where

P_2:

minimize $\quad L_2 = d_2^- + d_2^+$

subject to

$$c_2^t x + d_2^- - d_2^+ = h_2,$$
$$c_1^t x + d_1^- - d_1^+ = h_1,$$
$$d_1^- + d_1^+ = L_1^*,$$
$$Ax \le b,$$
$$d_i^- \ge 0,\ d_i^+ \ge 0, x \ge 0, \text{ and integer}, i = 1,2.$$

Letting $L_2^* = d_2^{-*} + d_2^{+*}$ to denote the integer optimal value of problem P_2, we can proceed to goal 3.

The general attainment problem P_j for goal j is written as

P_j :

minimize $\quad L_j = d_j^- + d_j^+$

subject to

$$c_i^t x + d_i^- - d_i^+ = h_i, \qquad 1 \le i \le j$$
$$d_i^- + d_i^+ = L_i^*, \qquad 1 \le i \le j-1$$
$$Ax \le b,$$
$$d_i^- \ge 0,\ d_i^+ \ge 0, x \ge 0, \text{ and integer}, \quad 1 \le i \le j$$

where d_i^- and d_i^+ are the underattainment and the overattainment, respectively, of the *i*th goal level and $d_i^- d_i^+ = 0$.

The integer optimal objective value of problem P_j, L_j^*, is the maximum degree of attainment for goal j subject to the maximum attainment of goals 1, 2,…, j−1. Notice that $L_j^* = 0$ if and only if goal j is attained.

Let x^* be the optimal integer solution of the integer attainment problem P_k associated with the minimum L_k^*; then the solution of the ILGP is given by x^*.

The procedure used to solve the ILGP can be summarized as follows.

4. SEQUENTIAL INTEGER GOAL ATTAINMENT ALGORITHM

Step 1. Formulate the ILGP corresponding to the (IMCDM) problem.

Step 2. Solve the integer attainment problem P_1 for goal 1 using Gomory's cutting-plane technique (Klein and Holm, 1978; 1979) and obtain L_1^*.

Step 3. Set $i = 2$.

Step 4. Using $L_1^*, L_2^*, \ldots, L_{i-1}^*$, solve the integer attainment problems P_i using the same cutting-plane technique used in step 2.

Let L_i^* denote the minimum.

Step 5. If $i \neq k$, set $i = i+1$ and go to step 4. Otherwise, go to step 6.

Step 6. Let $x^* = (x_1^*, x_2^*, \ldots, x_n^*)$ denote the integer solution(s) of the attainment problem P_k associated with the minimum L_k^*.

The optimal integer solution(s) of the ILGP is then given by x^*.

5. AN ILLUSTRATIVE EXAMPLE (CRISP CASE)

In this section, we consider the following integer multi-criteria decision-making problem with two objective functions:

(IMCDM):

Maximize $Z(x) = (z_1(x), z_2(x))$

subject to

$x \in X$

where

$$X = \left\{ x \in R^2 \;\middle|\; \begin{array}{l} x_1 + x_2 \le 5,\ -x_1 + x_2 \le 0,\ 6x_1 + 2x_2 \le 21,\ x_1, x_2 \ge 0 \\ \text{and integer} \end{array} \right\}$$

and

$$z_1(x) = 2x_1 + x_2$$
$$z_2(x) = x_1 + 2x_2.$$

Suppose that the decision maker specifies the first priority goal to be $z_1(x)$ and the second priority goal to be $z_2(x)$. Consequently, an equivalent integer linear goal program corresponding to the IMCDM problem can be written as follows:

(ILGP):

Goal 1: Achieve $2x_1 + x_2 = h_1$

Goal 2: Achieve $x_1 + 2x_2 = h_2$

subject to

$x \in X$

It is easy to see that the aspiration levels of the objectives $z_1(x)$ and $z_2(x)$ are $h_1(x) = 7$ and $h_2(x)$, respectively. The integer linear attainment problem associated with the first goal is written as

P_1:

minimize $L_1 = d_1^- + d_1^+$

subject to

$$2x_1 + x_2 + d_1^- - d_1^+ = 7$$
$$x_1 + x_2 \le 5$$
$$- x_1 + x_2 \le 0$$
$$6 x_1 + 2 x_2 \le 21$$
$$d_1^-,\ d_1^-,\ x_1,\ x_1 \ge 0 \quad \text{and} \quad \text{integer}$$

This problem can be solved using the following Gomory cuts, (see Klei and Holm, 1978;1979):

$$2x_1 + x_2 \le 7$$
$$x_1 + x_2 \le 4$$
$$x_1 \le 3$$

and the maximum degree of attainment of problem P_1 is $L_1^* = 0$, with an optimal integer solution $x^1 = (3,1)$ where $d_1^- = 0$ and $d_2^+ = 0$.

The attainment problem for goal 2 is equivalent to the integer optimization problem P_2, where

P_2:

minimize $L_2 = d_2^- + d_2^+$

subject to

$$x_1 + 2\bar{x}_2 + d_2^+ - d_2 = 6$$
$$2 x_1 + x_2 + d_1^- - d_1^+ = 7$$
$$d_1^- + d_1^+ = 0$$
$$x_1 + x_2 \le 5$$
$$- x_1 + x_2 \le 0$$
$$6 x_1 + 2 x_2 \le 21$$
$$d_i^-,\ d_i^+,\ x_1,\ x_2 \ge 0 \text{ and integer,}$$
$$i = 1, 2$$

The initial solution $x^1 = (3,1)$, $d_1^- = 0$ and $d_2^+ = 0$ yields a goal 2 value $x_1 + 2 x_2 = 5$.

The maximum degree of attainment of goal 2 is $L_2^* = 1$ with an optimal integer solution $x^2 = (3,1)$, where $d_2^- = 1$ and $d_2^+ = 0$. Therefore, the optimal integer solution of the ILGP is given by

$$x^* = (3, 1)$$

$$L_1^* = 0, \quad \text{with} \quad d_1^- = 0, \; d_1^+ = 0$$

$$L_2^* = 1, \quad \text{with} \quad d_2^- = 1, \; d_2^+ = 0$$

6. FUZZY INTEGER MULTI-CRITERIA DECISION-MAKING PROBEM (FIMCDM)

In this section, we begin by introducing the following fuzzy integer multi-criteria decision-making problem with fuzzy parameters in the right-hand side of the constraints as

$$(\text{FIMCDM})_{\tilde{v}}: \quad \text{Maximize } Z(x)$$

$$\text{subject to } x \in X(\tilde{v})$$

where

$$X(\tilde{v}) = \left\{ x \in R^n / g_r(x) \le \tilde{v}_r, \; (r = 1, 2, ..., m), \; x \ge 0 \text{ and integer} \right\}$$

and $Z : R^n \to R^k$, $Z(x) = (z_1(x), z_2(x), \ldots, z_k(x))$ is a vector-valued criterion with $z_i(x)$, $(i=1,2,..,k)$ are real-valued linear objective functions, $\tilde{v} = (\tilde{v}_1, \tilde{v}_2, ..., \tilde{v}_m)^t$ is a vector of fuzzy parameters, and R^n is the set of all ordered n-tuples of real numbers. Furthermore, the constraints functions $g_r(x)$, $(r = 1, 2, \ldots, m)$ are assumed to be linear.

Now, going back to problem $(\text{FIMCDM})_{\tilde{v}}$, we can write an associated fuzzy integer linear goal programming model $(\text{FILGP})_{\tilde{v}}$ consisting of k goals and having $\tilde{v} \in R^m$ a vector of fuzzy parameters in the right-hand side of the constraints. This model may be expressed as

$(\text{FILGP})_{\tilde{v}}$:

Achieve: $z_1(x) = h_1,$

$z_2(x) = h_2$

•
•
•

$z_k(x) = h_k$

and the constraints are given by

$$g_r(x) \leq \tilde{v}_r \text{ , } (r = 1, 2, \ldots, m)$$

$$x \geq 0 \text{ and integer}$$

where $h_1, h_2, \ldots, h_k$ are scalars and represent the aspiration levels associated with the objectives $z_1(x), z_2(x), \ldots, z_k(x)$, respectively.

7. FUZZY CONCEPTS

The fuzzy theory has been advanced by L.A. Zadeh at the University of California in 1965. This theory proposes a mathematical technique for dealing with imprecise concepts and problems that have many possible solutions. The concept of fuzzy mathematical programming on a general level was first proposed by Tanaka et al. (1974) in the framework of the fuzzy decision of Zadeh and Bellman (Zadeh, 1970).

For the development that follows, we introduce some defintions concerning trapeziodal fuzzy numbers and their membership functions, which come from (Dubois and Prade, 1980) ,and that will be used throughout this part. It should be noted that an equivalent approach can be used in the triangular fuzzy numbers case.

DEFINITION 3.

A real fuzzy number $\tilde{a}$ is a fuzzy subset from the real line R with membership function $\mu_{\tilde{a}}$ (a) that satisfies the following assumptions:

1. $\mu_{\tilde{a}}(a)$ is a continuous mapping from R to the closed interval [0, 1],
2. $\mu_{\tilde{a}}(a) = 0 \qquad \forall\, a \in (-\infty, a_1\,],$
3. $\mu_{\tilde{a}}(a)$ is strictly increasing and continuous on $[a_1, a_2]$,

4. $\mu_{\tilde{a}}(a) = 1 \quad \forall\, a \in [a_2, a_3]$,
5. $\mu_{\tilde{a}}(a)$ is strictly decreasing and continuous on $[a_3, a_4]$,
6. $\mu_{\tilde{a}}(a) = 0 \quad \forall\, a \in [a_4, +\infty)$.

where a_1, a_2, a_3, a_4 are real numbers and the fuzzy number $\tilde{a}$ is denoted by $\tilde{a} = [a_1, a_2, a_3, a_4]$.

DEFINITION 4.

The fuzzy number $\tilde{a}$ is a trapezoidal number, denoted by $[a_1, a_2, a_3, a_4]$, and its membership function $\mu_{\tilde{a}}(a)$ is given by (see Figure 1).

$$\mu_{\tilde{a}}(a) = \begin{cases} 0\,, & a \le a_1 \\ 1 - \left(\dfrac{a - a_2}{a_1 - a_2}\right)^2, & a_1 \le a \le a_2 \\ 1\,, & a_2 \le a \le a_3 \\ 1 - \left(\dfrac{a - a_3}{a_4 - a_3}\right)^2, & a_3 \le a \le a_4 \\ 0\,, & \text{otherwise.} \end{cases}$$

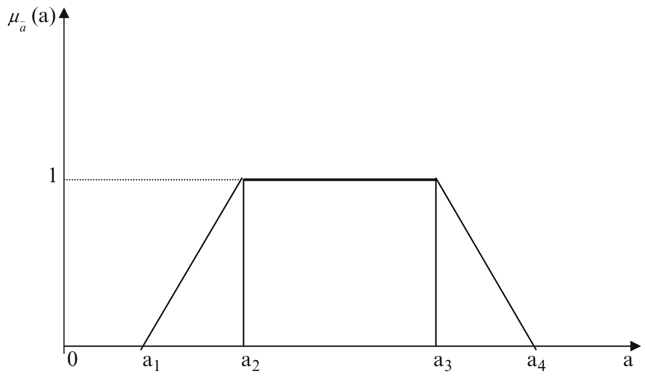

Figure 1. Membership function of a fuzzy number $\tilde{a}$

DEFINITION 5.

The α-level set of the fuzzy number $\tilde{a}$ is defined as the ordinary set $L_\alpha(\tilde{a})$ for which the degree of their membership function exceeds the level $\alpha \in [0, 1]$:

$$L_\alpha(\tilde{a}) = \left\{ a \in R \,\middle|\, \mu_{\tilde{a}}(a) \geq \alpha \right\}.$$

For a certain degree $\alpha^* \in [0, 1]$ with the corresponding α-level set of the fuzzy numbers $\tilde{v}_r$, problem $(\text{FILGP})_{\tilde{v}}$ can be understood as the following nonfuzzy integer linear goal programming model written as:

$(\text{FILGP})_v$:

Achieve: $z_1(x) = h_1$

$z_2(x) = h_2$

•
•
•

$z_k(x) = h_k$

subject to

$g_r(x) \leq v_r$, $(r = 1, 2, \ldots, m)$

$v_r \in L_\alpha(\tilde{v}_r)$, $(r = 1, 2, \ldots, m)$

$x \geq 0$ and integer

where $L_\alpha(\tilde{v}_r)$ is the α- level set of the fuzzy parameters $\tilde{v}_r$, $(r = 1, 2, \ldots, m)$.

We now rewrite problem $(\text{FILGP})_v$ above in the following equivalent form:

$(ILGP)_v$:

Achieve: $z_1(x) = h_1$

$z_2(x) = h_2$

•
•
•

$z_k(x) = h_k$

subject to

$$g_r(x) \leq \nu_r , \quad (r = 1, 2, \ldots, m)$$

$$n_r^{(0)} \leq \nu_r \leq N_r^{(0)}, (r = 1, 2, \ldots, m)$$

$$x \geq 0 \text{ and integer}$$

It should be noted that the constraint $\nu_r \leq L_\alpha(\tilde{\nu}_r)$, ($r = 1, 2, \ldots, m$), has been replaced by the equivalent constraint $n_r^{(0)} \leq \nu_r \leq N_r^{(0)}$, ($r = 1, 2, \ldots, m$), where $n_r^{(0)}$ and $N_r^{(0)}$ are lower and upper bounds on ν_r.

Taking into account restrictions $g_r(x) \leq \overset{*}{\nu_r}$, ($r = 1, 2, \ldots, m$) and for the purpose of solving the integer linear goal program $(ILGP)_\nu$ at $\nu_r = \nu_r^* = N_r^{(0)}$, ($r = 1,2,\ldots,m$) for a certain degree $\alpha = \alpha^* \in [0, 1]$, we use the iterative approach developed in Dauer and Rrueger (1977) together with the Gromory cuts shown in Klein and Holm (1978, 1979). First, we solve the following integer linear optimization problem associated with the first goal, viz:

$P_1(\nu_r^*)$:

$$\text{Minimize} \quad L_1 = d_1^- + d_1^+$$

subject to

$$z_1(x) + d_1^- - d_1^+ = h_1$$

$$g_r(x) \leq \overset{*}{\nu_r}, (r = 1, 2, \ldots, m)$$

$$d_1^- \geq 0, \ d_1^+ \geq 0, x \geq 0 \text{ and integer}$$

where d_1^- and d_1^+ are the underattainment and the overattainment, respectively, of the first goal where $d_1^- \, d_1^+ = 0$.

Suppose this problem has integer optimal value $L_1^* = d_1^{-*} + d_1^{+*}$ with at least one value d_1^{-*} or d_1^{+*} nonzero.

Now, the attainment problem for goal 2 is equivalent to the integer optimization $P_2(\nu_r^*)$, where

$P_2(\nu_r^*)$:

$$\text{Minimize} \quad L_2 = d_2^- + d_2^+$$

subject to

$$z_2(x) + d_2^- - d_2^+ = h_2$$

$$z_1(x) + d_1^- - d_1^+ = h_1$$

$$d_1^- + d_1^+ = L_1^*$$

$$g_r(\mathrm{x}) \le \nu_r^* , \ (r = 1, 2, \ldots, m)$$

$$d_i^- \ge 0 , \ d_i^+ \ge 0 , \ \mathrm{x} \ge 0 \text{ and integer}, \ (i = 1,2)$$

Letting $L_2^* = d_2^{-*} + d_2^{+*}$ denotes the integer optimal value of $P_2(\nu_r^*)$, we can proceed to goal 3.

The general attainment problem $P_j(\nu_r^*)$ for goal j is written as

$P_j(\nu_r^*)$:

Minimize $L_j = d_j^- + d_j^+$

subject to

$$z_i(x) + d_i^- - d_i^+ = h_i, \ (1 \le i \le j)$$

$$d_i^- + d_i^+ = L_i^*, \ (1 \le i \le j\text{-}1)$$

$$g_r(\mathrm{x}) \le \nu_r^* , \ (r = 1, 2, \ldots, m)$$

$$d_i^- \ge 0 , \ d_i^+ \ge 0 , \ \mathrm{x} \ge 0 \text{ and integer}, \ (1 \le i \le j)$$

where d_i^- and d_i^+ are the underattainment and the overattainment, respectively, of the ith goal level and $d_i^- \, d_i^+ = 0$.

The integer objective value of $P_j(\nu_r^*)$, L_j^*, is the maximum degree of attainment for goal j subject to the maximum attainment of goals 1, 2,…, j–1. Notice that $L_j^* = 0$ if and only if goal j is attained.

Let x^* be the optimal integer solution of the integer attainment problem $P_j(\nu_r^*)$ associated with the minimum L_j^*, then the solution of the integer goal program $(ILGP)_\nu$ is given by x^* with $\alpha = \alpha^* \in [0, 1]$.

8. AN ITERATIVE GOAL PROGRAMMING APPROACH FOR SOLVING FIMCDM

Now, we develop a solution algorithm to solve the fuzzy integer linear goal program $(FILGP)_{\tilde{v}}$ .The outline of this algorithm is as follows (**Alg-II)**:

Step 0. Set $\alpha = {}^{v}\alpha^{*} = 0$.

Step 1. Determine the points (a_1, a_2, a_3, a_4) for each fuzzy parameter $\tilde{v}_r$, (r = 1, 2, …, m) in program $(FILGP)_{\tilde{v}}$ with the corresponding membership function $\mu_{\tilde{a}}(\tilde{v}) \geq \alpha^{*}$ for the vector of fuzzy parameters $\tilde{v} = (\tilde{v}_1, \tilde{v}_2, ..., \tilde{v}_m)^t$.

Step 2. Convert program $(FILGP)_{\tilde{v}}$ into the nonfuzzy integer linear goal program $(ILGP)_v$.

Step 3. Choose $v_r = v_r^{*} = N_r^{(0)}$, ($r$ = 1, 2,…, m) and solve problem $P_1(v_r^{*})$ using Gomory's cutting- plane method (Klein and Holm, 1978, 1979) and obtain L_1^{*}.

Step 4. Set j =2.

Step 5. Using L_1^{*}, L_2^{*},…, L_{j-1}^{*}, solve $P_j(v_r^{*})$ using the same Gomory's cutting-plane method used in step 3.

Let L_j^{*} denotes the minimum.

Step 6. If $j \neq k$, set $j = j + 1$ and go to step 5. Otherwise, go to Step 7.

Step 7. Let x^{*} denotes the optimal integer solution of problem $P_j(v_r^{*})$ associated with the minimum L_j^{*}.

Step 8. Set $\alpha = (\alpha^{*} + \text{step}) \in [0, 1]$, and go to Step 1.

Step 9. Repeat again the above procedure until the interval [0, 1] is fully exhausted. Then stop.

9. AN ILLUSTRATIVE EXAMPLE (FUZZY CASE)

Consider the following integer linear goal program involving fuzzy parameters $(\tilde{v}_1, \tilde{v}_2, \tilde{v}_3)$ in the right-hand side of the constraints:

$(FILGP)_{\tilde{v}}$:

goal 1: Achieve $2x_1 + x_2 = h_1$

goal 2: Achieve $x_1 + 2x_2 = h_2$

subject to

$$x_1 + x_2 \leq \tilde{v}_1$$

$$-x_1 + x_2 \leq \tilde{v}_2$$

$$6x_1 + 2x_2 \leq \tilde{v}_3$$
$$x_1 \geq 0, x_2 \geq 0 \text{ and integers.}$$

Assume that the membership function corresponding to the fuzzy parameters is in the form:

$\mu_{\tilde{a}}(a) =$

$$\mu_{\tilde{a}}(a) = \begin{cases} 0, & a \leq a_1 \\ 1 - \left(\dfrac{a - a_2}{a_1 - a_2}\right)^2, & a_1 \leq a \leq a_2 \\ 1, & a_2 \leq a \leq a_3 \\ 1 - \left(\dfrac{a - a_3}{a_4 - a_3}\right)^2, & a_3 \leq a \leq a_4 \\ 0, & a \geq a_4 \end{cases}$$

where $\tilde{a}$ corresponds to each $\tilde{v}_i$, (i = 1, 2, 3). In addition, we assume also that the fuzzy unmbers are given by the following values:

$$\tilde{v}_1 = (2, 4, 6, 8),\ \tilde{v}_2 = (0, 3, 5, 7),\ \tilde{v}_3 = (18, 20, 22, 24).$$

Setting $\alpha = \alpha^* = 0$, then we get

$$2 \leq \tilde{v}_1 \leq 8,\ 0 \leq \tilde{v}_2 \leq 7,\ 18 \leq \tilde{v}_3 \leq 24.$$

By choosing $v^* = (v_1^*, v_2^*, v_3^*) = (8, 7, 24)$, then the aspiration levels of the goals have been found $h_1 = 10$ and $h_2 = 15$, respectively.

The integer optimization problem associated with the first goal is

$P_1(v_r^*)$:

Minimize $L_1 = d_1^- + d_1^+$

subject to

$$2x_1 + x_2 + d_1^- - d_1^+ = 10$$
$$x_1 + x_2 \leq 8$$
$$-x_1 + x_2 \leq 7$$
$$6x_1 + 2x_2 \leq 24$$
$$d_1^- \geq 0,\ d_1^+ \geq 0,\ x_1 \geq 0,\ x_2 \geq 0 \text{ and integers.}$$

The maximum degree of attainment of problem $P_1(v_r^*)$ is $L_1^* = 0$ with the optimal integer solution:

$$x^1 = (2, 6) \text{ and } d_1^- = 0, \quad d_1^+ = 0.$$

The attainment problem for goal 2 is equivalent to the integer optimization problem $P_2(v_r^*)$ where

$P_2(v_r^*)$:

Minimize $L_2 = d_2^- + d_2^+$

subject to

$$x_1 + 2x_2 + d_2^- - d_2^+ = 15$$
$$2x_1 + x_2 + d_1^- - d_1^+ = 10$$
$$d_1^- + d_1^+ = 0$$
$$x_1 + x_2 \le 8$$
$$-x_1 + x_2 \le 7$$
$$6x_1 + 2x_2 \le 24$$
$$d_i^- \ge 0, \; d_i^+ \ge 0, \; x_1 \ge 0, \; x_2 \ge 0, \text{ and integers } (i = 1, 2).$$

The maximum degree of attainment of goal 2 is $L_2^* = 1$ with the optimal integer solution:

$$x^2 = (2, 6) \text{ and } d_2^- = 1, \; d_2^+ = 0$$

Therefore, the optimal integer solution of the original integer linear goal program is:

$$x^* = (2, 6)$$

$$L_1^* = 0 \quad \text{with} \quad d_1^{-*} = 0, \; d_1^{+*} = 0$$

$$L_2^* = 1 \quad \text{with} \; d_2^{-*} = 1, \; d_2^{+*} = 0$$

with the corresponding used Gomory cut: $x_2 \le 7$.

On the other hand, setting $\alpha = \alpha^* = 1$, we get:

$$4 \le \tilde{v}_1 \le 6, \; 3 \le \tilde{v}_2 \le 5, \; 20 \le \tilde{v}_3 \le 22.$$

Choosing $v^* = (v_1^*, v_2^*, v_3^*) = (6, 5, 22)$, then the optimal integer solution of the original program has been found:

$$x^* = (2, 4)$$

$$L_1^* = 0 \quad \text{with} \quad d_1^{-*} = 0,\ d_1^{+*} = 0$$

$$L_2^* = 1 \quad \text{with} \quad d_2^{-*} = 1,\ d_2^{+*} = 0$$

with the corresponding used Gomory cut: $3x_1 + x_2 \leq 10$.

Remark. It should be noted that a systematic variation of $\alpha \in [0, 1]$ will yield a new optimal integer solution to the integer linear goal program $(\text{FILGP})_{\tilde{v}}$

10. CONCLUSION

Since goal programming now encompasses any linear, integer, zero−one, or nonlinear multi-objective problem (for which preemptive priorities may be established), the field of applications is wide open. The recent increase in interest in this area has already led to a large number of and wide variety of actual and proposed applications. In this chapter, we have given numerical examples for the IMCDM problem and the FIMCDM. Fuzzy goal programming has many opportunities to develop new approaches to it.

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GREY FUZZY MULTI-OBJECTIVE OPTIMIZATION

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Abstract: This chapter provides a description of grey fuzzy multi-objective optimization. A prerequisite background on grey systems, along with preliminary definitions is provided. Formulation of the grey fuzzy optimization starting with a general fuzzy optimization problem is discussed. Extension of the grey fuzzy optimization with the acceptability index to include multiple objectives is presented. Application of the grey fuzzy multi-objective optimization is demonstrated with the problem of waste load allocation in the field of environmental engineering.

Key words: Grey fuzzy, fuzzy optimization, waste load allocation

1. INTRODUCTION

Uncertainties in decision models may stem from a number of factors such as randomness of input parameters, imprecision in management goals, inappropriateness in model selection leading to scenario uncertainty, broad range of possible alternative formulations, and uncertainties in input parameters due to inadequate of data. Uncertainty due to randomness of input parameters may be modeled using the probability theory when adequate data are available to satisfactorily estimate the probability distributions of the parameters. Uncertainties due to imprecision in the management problem, on the other hand, are modeled using the fuzzy sets theory, by appropriately constructing membership functions for the fuzzy or imprecisely defined goals and constraints. In addition, model parameters in most optimization problems need to be addressed as *grey parameters*, due to inadequate data for an accurate estimation but with known extreme

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bounds of the parameter values. Such grey uncertainty, with partially known and partially unknown characteristics cannot be effectively modeled by probabilistic or fuzzy logic approach because of inadequacy of data to estimate probability distribution and lack of information to precisely define the membership functions.

Interval programming (IP) provides a methodology for modeling inexactness in parameters (e.g., left-hand side model coefficients and right-hand side stipulations of constraints) of an optimization model, by considering them as interval numbers (Dantzig, 1963; Jaulin et al., 2001; Moore, 1979; Tong, 1994). A reason for the lack of many useful applications of interval programming is that the solution procedure is too complicated and time consuming (Dantzig, 1963; Jansson, 1988; Moore, 1979). Grey optimization (Huang et al., 1992, 1995, 2001) of grey systems theory (Deng, 1982) offers methods for incorporating uncertainties in model parameters directly in an optimization framework avoiding huge data requirement and mathematical complicacy. The grey uncertainty or inexactness of model parameters can be addressed by representing them as interval grey numbers, instead of deterministic real numbers. An interval grey number is a closed interval with known lower and upper bounds but unknown distribution information (Huang et al., 1992; 1995; Liu and Lin, 1998). Both interval and grey programming techniques are used for determining interval-valued solutions of an optimization model in which coefficients in objective function, left-hand side model coefficients, and right-hand side stipulations of constraints are represented by closed intervals. A basic difference between interval programming and grey programming lies in their solution procedures. The primary goal of a grey optimization model is to determine the two extreme values of the optimal interval-valued decision variables in most adverse and favorable conditions (Huang et al., 1995, Karmakar and Mujumdar, 2005b). Huang et al. (1992, 1993, 1995), and Chen and Huang (2001) have presented a few such novel efforts to find the solutions from grey linear, integer and quadratic programming models.

A number of research contributions are available in the literature that deal with uncertainty due to imprecision, fuzziness, or vagueness, where fuzzy sets theory (Zadeh, 1965) is the only tool used to address such uncertainty. The imprecision associated with management goals and constraints is quantified using membership functions, which are normally represented by a geometric shape that defines how each point in the input space is mapped to a membership value between 0 and 1. For example, to account for the imprecision in the standards for determining a failure of water quality, occurrence of failure is treated as a fuzzy event (Mujumdar and Sasikumar, 2002); a fuzzy set of low water quality maps all water quality levels to “low water quality,” and its membership function denotes the degree to which the water quality is low. The membership functions represent the perceptions of the decision makers and other stakeholders in most decision-making problems. The boundaries (Ross, 1995) of the membership functions—or, the membership parameters—are assumed fixed, and values to the parameters are assigned based on experience and judgment. As the model solution is likely to vary considerably with change in the membership functions, uncertainty in the boundaries and shape of the membership functions should also be addressed in a fuzzy optimization model. Some studies address modeling of uncertainty in the values of membership parameters by considering the membership function itself as fuzzy. Type−2 fuzzy sets (Karnik and Mendel, 2001; Mizumoto and Tanaka, 1976; Mendel 2001; Zadeh, 1975), interval-valued fuzzy sets (Chiang, 2001; Turksen and Bilgiç, 1996), and grey fuzzy optimization (Chang et al., 1996; 1997; Karmakar and Mujumdar, 2005a, b; Maqsood et al., 2005) are some examples of attempts to model such uncertainty. Mathematically grey fuzzy optimization is the simplest way to model the uncertainty in membership parameters. In this approach, the membership parameters are considered as interval grey numbers. A set of optimal interval-valued decision variables are obtained as solution, corresponding to a maximized interval-valued goal fulfillment level, whereas conventional fuzzy optimization model gives only a single set of optimal decision variables corresponding to the maximum goal fulfillment level (Chang et al., 1997; Karmakar and Mujumdar, 2005b; Zhang and Huang, 1994). This feature of the solution from a grey fuzzy optimization model imparts flexibility in decision making. The width of the interval-valued optimal decision variables plays an important role in the grey fuzzy optimization model, as more width in the optimal values of decision variables implies a wider choice to the decision-makers. The width of the optimal interval-valued goal fulfillment level, on the other hand, implies the system uncertainty, which should be minimized in a grey fuzzy optimization model. Grey fuzzy multi-objective optimization is a potential approach to maximize the width of the interval-valued optimal decision variables for providing latitude in decision making and to minimize the width of the goal fulfillment level for reducing the system uncertainty (Karmakar and Mujumdar, 2005a). The discussion in this chapter is restricted to grey fuzzy optimization techniques mainly focusing on grey fuzzy multi-objective optimization. As a prerequisite, a brief overview of the grey systems theory is provided first.

2. GREY SYSTEMS THEORY

Grey systems theory was first proposed by Deng (1982). Concepts of grey systems are different from those of probability and statistics, which address problems with samples of a reasonable size, and also different from those of fuzzy mathematics, which deal with problems with cognitive uncertainty. Table 1 presents some features of grey systems theory, probability theory, and fuzzy mathematics.

Table 1. Features of Grey Systems Theory, Probability Theory and Fuzzy Mathematics (Deng 1982, Liu and Lin, 1998)

	Grey systems theory	Probability theory	Fuzzy mathematics
Intention	Small sample uncertainty	Large sample uncertainty	Cognitive uncertainty
Foundation	Information coverage	Probability distribution	Function of affiliation
Characteristics	Few data points	Large number of data points	Experience
Requirement	Any distribution	Probability distribution	Membership function
Objective	Laws of reality	Laws of statistics	Cognitive expressions

A grey system is a system other than a white system (system with completely known information) and a black system (system with completely unknown information), and thus it has partially known and partially unknown characteristics. Table 2 shows the major characteristics of the white, black, and grey systems.

Table 2. Characteristics of White, Black and Grey Systems (Liu and Lin 1998)

	White system	Black system	Grey system
Information	Known	Unknown	Incomplete
Appearance	Bright	Dark	Grey
Property	Order	Chaos	Complexity
Attitude	Surety	Indulgence	Tolerance
Conclusion	Unique solution	No result	Multiple solution

Most processes of interest in decision problems are in the grey stage due to the inadequate and/or fuzzy information. Grey fuzzy optimization provides a useful tool for decision making addressing such uncertainties. As a background to formulation of a grey fuzzy optimization model, we first provide a few definitions related to interval analysis (Moore, 1979) and grey systems theory (Huang et al., 1995; Liu and Lin, 1998).

DEFINITION 1.

A "grey number" is such a number whose exact value is unknown, but a range within which the value lies is known [Liu and Lin 1998]. Let x denote a closed and bounded set of real numbers. A grey number ($x^{\pm}$) is defined as an interval with known lower (x^{-}) and upper (x^{+}) bounds but with unknown distribution information for x (Huang et al., 1997).

$$x^{\pm} = [x^{-}, x^{+}] = [t \in x \mid x^{-} \leq t \leq x^{+}] \quad (1)$$

$x^{\pm}$ becomes a "deterministic number" or "white number" when, $x^{\pm} = x^{-} = x^{+}$. When $x^{\pm} = [x^{-}, x^{+}] = (-\infty, +\infty)$ or $x^{\pm} = [x_1^{\pm}, x_2^{\pm}]$, that is, $x^{\pm}$ has neither lower limit nor upper limit, or the lower and the upper limits are all grey numbers, $x^{\pm}$ is called a "black number." An "interval number" or "interval grey number" ($x^{\pm} = [x^{-}, x^{+}]$) is one among several classes of grey numbers.

DEFINITION 2.

The "whitened value" of a grey number, $x^{\pm}$, is defined as a deterministic number with its value lying between the upper and lower bounds of $x^{\pm}$; i.e., $x^{-} \leq x_v \leq x^{+}$, when x_v is a whitened value of $x^{\pm}$. For a general interval grey number,

$$\mathrm{x_v} = x^{-}\sigma + (1-\sigma)\, x^{+},\ \sigma \in [0, 1] \quad (2)$$

is called "equal weight whitenization" (Liu and Lin, 1998). "σ" is a weight factor that can take any value between 0 and 1.

DEFINITION 3.

In an equal weight whitenization, the whitened value obtained, when taking σ = ½, is called an "equal weight mean whitenization" or "Whitened Mid Value" (WMV). Therefore, WMV of $x^{\pm}$ is written as (Liu and Lin, 1998):

$$x_{\mathrm{m}} = ½\,(x^{-} + x^{+}) \quad (3)$$

DEFINITION 4.

The "grey degree" is a measure, useful for quantitatively evaluating the quality of input or output uncertain information for mathematical models

(Huang et al., 1997). The "grey degree" of an interval grey number is defined as its width [$x_\omega = (x^+ - x^-)$] divided by its WMV [$x_m = ½ (x^- + x^+)$] (Huang et al., 1995) and is expressed in percentage (%) as follows:

$$Gd(x^{\pm}) = (x_\omega / x_m) \times 100\ \% \tag{4}$$

where $Gd(x^{\pm})$ is the grey degree of $x^{\pm}$. Solutions (model outputs) with considerably high grey degree have high width (x_ω) of output variables, which are considered as less useful and of poor quality for decision making. As the grey degree of objective function of an optimization model decreases, implying decreasing system uncertainties, the usefulness of the grey model increases.

DEFINITION 5.

A "grey system" is defined as a system containing information presented as grey numbers [Huang et al., 1995].

DEFINITION 6.

Let $* \in \{+, -, \times, \div\}$ be a binary operation on grey numbers. Therefore, the operations can be expressed as (Huang et al., 1995; Ishibuchi and Tanaka, 1990)

$$x^{\pm} * y^{\pm} = [\min (x * y), \max (x * y)],\ x^- \le x \le x^+, y^- \le y \le y^+ \tag{5}$$

For different binary operations:

$$x^{\pm} + y^{\pm} = [(x^- + y^-), (x^+ + y^+)] \tag{6}$$

$$x^{\pm} - y^{\pm} = [(x^- - y^+), (x^+ - y^-)] \tag{7}$$

$$x^{\pm} \times y^{\pm} = [\min (x \times y), \max (x \times y)] \tag{8}$$

$$x^{\pm} \div y^{\pm} = [\min (x \div y), \max (x \div y)], \text{ when } 0 \notin y^{\pm} \tag{9}$$

DEFINITION 7.

A "general mathematical model" of grey linear programming is as follows (Huang et al., 1992):

$$\text{Maximize } f^{\pm} = c^{\pm} x^{\pm} \tag{10}$$

subject to

$$A^{\pm} x^{\pm} \leq b^{\pm} \tag{11}$$

$$x^{\pm} \geq 0 \tag{12}$$

where $c^{\pm} = [c_1^{\pm}, c_2^{\pm}, \ldots.., c_n^{\pm}]$; $x^{\pm} = [x_1^{\pm}, x_2^{\pm}, \ldots.., x_n^{\pm}]^T$; $b^{\pm} = [b_1^{\pm}, b_2^{\pm}, \ldots.., b_m^{\pm}]^T$; $A^{\pm} = \{a_{ij}^{\pm}\}$, $\forall\ i = 1, 2, \ldots\ldots, m$, and $j = 1, 2, \ldots\ldots, n$. Since interval grey parameters exist in the objective function and constraints, the optimal solutions of grey linear programming model are $f^{\pm} = [\hat{f}^-, \hat{f}^+]$, and $x^{\pm} = [\hat{x}_1^{\pm}, \hat{x}_2^{\pm}, \ldots\ldots, \hat{x}_n^{\pm}]$, where $\hat{x}_j^{\pm} = [\hat{x}_j^-, \hat{x}_j^+]$, $\forall\ j = 1, 2, \ldots\ldots, n$. The primary goal of a grey optimization model is to determine the two extreme values of the optimal interval-valued decision variables, $\hat{x}^{\pm}$, in most adverse and favorable conditions considering the appropriate extreme bounds of the pre-specified parameters in the model constraints, i.e., $c^{\pm}$, $A^{\pm}$, $b^{\pm}$. It is to be noted that the grey optimization model does not include the situation when a model parameter expressed as an interval grey number, contains a zero with the two bounds having different signs (e.g., $b_i^{\pm} = [-b_i^-, +b_i^+]$, where $b_i > 0$). Details of the solution algorithm for a grey optimization problem may be found in Huang et al. (1994, 1995).

3. CONCEPT OF A GREY FUZZY DECISION

Fuzzy optimization (Zimmermann, 1978) is an application of fuzzy sets theory that determines optimal solution in the presence of imprecisely defined goals and constraints. Bellman and Zadeh (1970) proposed a broad definition of the fuzzy decision as a confluence of fuzzy goals and fuzzy constraints, which is the basis of fuzzy optimization. Noting that the decision space is defined by the intersection of different fuzzy goals, the fuzzy decision (D) is written as follows:

$$D = F_1 \cap F_2 \tag{13}$$

where fuzzy sets F_1 and F_2 represent the two fuzzy goals. The membership function of the fuzzy decision (D) is given by

$$\mu_D(x) = \lambda = \min\ [\mu_{F_1}(x),\ \mu_{F_2}(x)]. \tag{14}$$

"λ" is the measuring variable corresponding to the membership function of fuzzy decision (D), which reflects the degree of fulfillment of the system goals. A terminology of "goal fulfillment level" is used throughout the chapter to represent "λ." In the concept of fuzzy decision (D) as described by Eq. (13), the arguments of F_1 and F_2 are deterministic real numbers (x). When the goals $F_1^{\pm}$ and $F_2^{\pm}$ are imprecise fuzzy goals or grey fuzzy goals, i.e., the uncertain membership parameters are considered as interval grey numbers and corresponding arguments are interval grey numbers ($x^{\pm}$), the fuzzy decision leads to a "grey fuzzy decision ($D^{\pm}$)" (Karmakar and Mujumdar, 2005a, 2000). This terminology is earlier used by Luo et al. (1999) to define a "grey fuzzy motion decision" combining grey prediction and fuzzy logic control theories. The notion of "grey fuzzy decision" presented in this chapter is different from that used by Luo et al. (1999). Here grey fuzzy decision represents the fuzzy decision resulting from the imprecise membership functions, where the membership parameters are expressed as interval grey numbers (Karmakar and Mujumdar, 2005b). Figure 1 illustrates the concept of grey fuzzy decision considering the confluence of two imprecise membership functions for grey fuzzy goals, $F_1^{\pm}$ and $F_2^{\pm}$. Considering "logical and," corresponding to the "set theoretic intersection" as an aggregation operator, the grey fuzzy decision is determined. In Figure 1, the decision space is is defined by the lower and upper boundaries A "FNGH" and A"ECMC'HH," respectively.

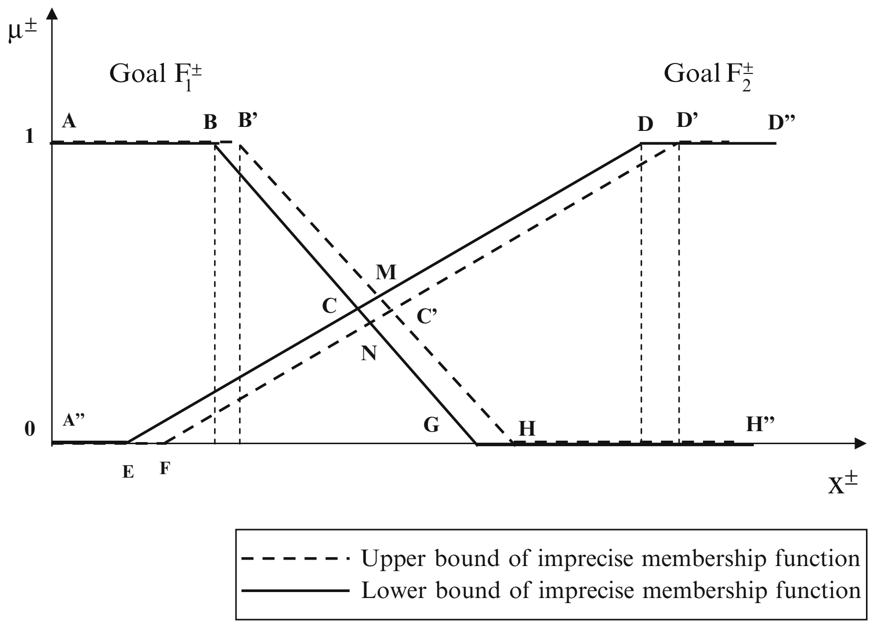

Figure 1. Concept of grey fuzzy decision

The solutions $\hat{x}^{\pm}$, corresponding to the maximum value of the membership function of the resulting grey fuzzy decision ($D^{\pm}$) is an interval in the space CMC'N (Figure 1). Mathematically the grey fuzzy decision ($D^{\pm}$) for $F_1^{\pm}$ and $F_2^{\pm}$ can be defined with the imprecise membership function (Karmakar and Mujumdar, 2005a):

$$\mu^{-}_{D^{\pm}}(x^{\pm}) = \lambda^{-} = \min[\min\{\mu^{-}_{F_1^{\pm}}(x^{-}), \mu^{-}_{F_2^{\pm}}(x^{-})\}; \min\{\mu^{-}_{F_1^{\pm}}(x^{+}), \mu^{-}_{F_2^{\pm}}(x^{+})\}] \quad (15)$$

$$\mu^{+}_{D^{\pm}}(x^{\pm}) = \lambda^{+} = \max[\min\{\mu^{+}_{F_1^{\pm}}(x^{-}), \mu^{+}_{F_2^{\pm}}(x^{-})\}; \min\{\mu^{+}_{F_1^{\pm}}(x^{+}), \mu^{+}_{F_2^{\pm}}(x^{+})\}] \quad (16)$$

where $\mu^{-}_{D^{\pm}}(x^{\pm})$ and $\mu^{+}_{D^{\pm}}(x^{\pm})$ are lower and upper bounds of the imprecise membership functions for an interval $[x^{-}, x^{+}]$, respectively. Eqs. (15) and (16) are valid for all combinations of imprecise membership functions (i.e., non-increasing, nondecreasing, or a combination of the two). Eqs. (15) and (16) are readily extendible to any number of imprecise goals.

4. GREY FUZZY OPTIMIZATION

The grey fuzzy optimization technique is based on the concept of grey fuzzy decision. Determination of an appropriate deterministic equivalent of the grey fuzzy optimization model is still a potential research area. Following the notations used by Zimmermann (1985), a generalized fuzzy optimization model may be written as

$$\begin{aligned} &\text{Maximize } (\lambda) \\ &\text{subject to } \left(\frac{d_i' - B_i x}{d_i' - d_i}\right)^{\gamma_i} \geq \lambda \quad \forall i, \ for \ i = 1, \ldots, m \\ &0 \leq \lambda \leq 1 \\ &x \geq 0, \ x \in \Re^n \end{aligned} \quad (17\text{–}20)$$

The solution of model (17)–(20) gives the optimal values of x satisfying all m numbers of fuzzy goals, expressed by the constraint $B_i x \lesssim d_i$ for $i = 1, \ldots, m$;

with maximized level of goal fulfillment, $\hat{\lambda}$. Here "$\lesssim$" is "fuzzified" version of "≤" and has the linguistic interpretation "essentially less than or equal to." Constraint (18) denotes the ith membership function, $\mu_i(x)$, interpreted as the degree to which x fulfills the fuzzy goal, where B_i and d_i denote the i-th row of B and d, respectively. The exponent γ_i is a nonzero positive real number. Assignment of numerical value to this exponent is subject to the desired shape of the membership functions. *A* value of $\gamma_i = 1$ leads to the linear membership function. The value of $\mu_i(x)$ should be 0 if the set of constraints are strongly violated and 1 if they are well satisfied. $\mu_i(x)$ should increase monotonically from 0 to 1. Figure 2 shows a linear membership function of B_ix, where two membership parameters are at the desirable (d_i) level and maximum permissible [$d_i' = (d_i + p_i)$] level, fixed by the decision maker.

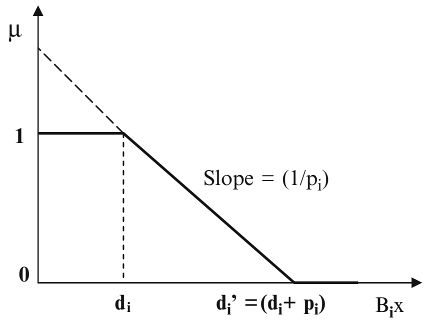

Figure 2. Linear membership function

The value of p_i is uncertain and a subjectively chosen constant of the admissible violation for i-th fuzzy goal. The membership function, $\mu_i(x)$, results in an imprecise membership function, $\mu_i^{\pm}(x^{\pm})$, when the uncertainty in the value of p_i is considered as interval grey number ($p_i^{\pm}$). The current discussion focuses on modeling the uncertainty in membership parameters considering the boundaries of the membership functions as interval grey numbers, which results in the value of p_i as an interval grey number. Figure 3 shows a linear imprecise membership function where the uncertain value of $p_i^{\pm}$ is expressed as $(d_i'^{\pm} - d_i^{\pm})$, and extreme bounds are presented as $p_i^{-} = (d_i'^{-} - d_i^{+})$ and $p_i^{+} = (d_i'^{+} - d_i^{-})$.

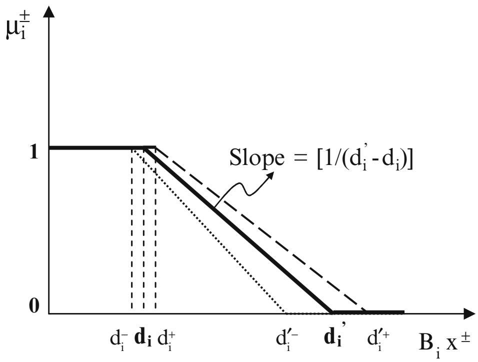

Figure 3. Linear imprecise membership function

Mathematically the imprecise membership function $\mu_i^{\pm}(x^{\pm})$ can be expressed as

$$\mu_i^{\pm}(x^{\pm}) = \begin{cases} 1 & \text{if} \quad B_i\, x^{+} < d_i^{-} \\ \left(\dfrac{d_i'^{\pm} - B_i\, x^{\pm}}{d_i'^{\pm} - d_i^{\pm}}\right)^{\gamma_i} & \text{if} \quad d_i^{\pm} \le B_i\, x^{\pm} \le d_i'^{\pm} \ \text{ for } i = 1, \ldots, m \\ 0 & \text{if} \quad B_i\, x^{-} > d_i'^{+} \end{cases} \qquad (21\text{–}23)$$

Similar to the max−min formulation for fuzzy optimization by Zimmermann (1978), the grey fuzzy optimization model can be represented as

$$\text{Find} \ \ (x^{\pm})$$

$$\max_{\substack{x^{+} \ge 0, \\ x^{-} \ge 0}} \min_{i} \left(\frac{d_i'^{\pm} - B_i\, x^{\pm}}{d_i'^{\pm} - d_i^{\pm}}\right)^{\gamma_i} \qquad (24\text{–}25)$$

In the grey fuzzy optimization model, the input vector B_i can also be uncertain, depending on the particular problem being solved. A more generalized form of the grey fuzzy optimization model can be obtained by considering B_i as an interval grey number ($B_i^{\pm}$). Similar to fuzzy optimization model (17)−(20), a generalized form of the grey fuzzy optimization model may be written as

$$\text{Maximize } (\lambda^{\pm})$$
$$\text{subject to } \left(\frac{d_i'^{\pm} - (B_i\, x)^{\pm}}{d_i'^{\pm} - d_i^{\pm}}\right)^{\gamma_i} \geq \lambda^{\pm} \quad \forall i \qquad (26\text{–}29)$$
$$0 \leq \lambda^{\pm} \leq 1$$
$$x^{\pm} \geq 0$$

where B_{i} is an interval grey number that results in the value of $(B_{\text{i}}x)^{\pm}$ [i.e., $(B_{\text{i}}^{\pm} \times x^{\pm})^{\pm}$] as an interval grey number, following Eq. (8) in Definition 6. A typical confluence of two non increasing linear imprecise membership functions as described in Eqs. (21)−(23), when i = 1, 2 and γ_{i} = 1, are shown in Figure 4. In Figure 4, the lower and upper boundaries of grey fuzzy decision are ABD'F'FG and ABCDEFG, respectively.

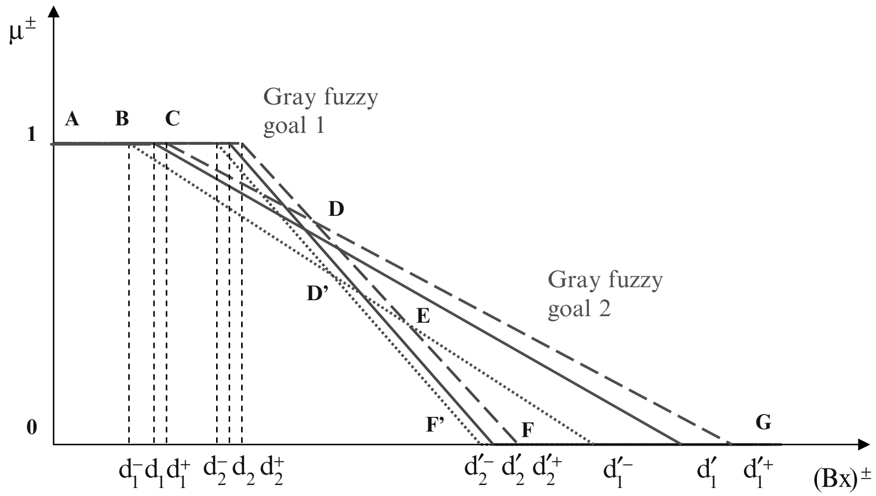

Figure 4. Confluence of gray fuzzy goals

To obtain the two extreme values of optimum goal fulfillment level ($\hat{\lambda}^+$ and $\hat{\lambda}^-$), that provide solutions for two extreme cases encompassing all intermediate possibilities, the deterministic equivalent of the grey fuzzy optimization model [Eqs. (26)−(29)] is divided into two sub-models as:

Sub-model 1

$$\text{Maximize} \;\; (\lambda^{+})$$
$$\text{subject to}$$
$$\left(\frac{d_{\mathrm{i}}'^{+} - (B_i\, x)^{-}}{d_{\mathrm{i}}'^{-} - d_i^{+}}\right)^{\gamma_{\mathrm{i}}} \geq \lambda^{+} \quad \forall i \tag{30–33}$$
$$0 \leq \lambda^{+} \leq 1$$
$$x^{-} \geq 0$$

Sub-model 2

$$\text{Maximize} \;\; (\lambda^{-})$$
$$\text{subject to}$$
$$\left(\frac{d_i'^{-} - (B_i\, x)^{+}}{d_i'^{+} - d_i^{-}}\right)^{\gamma_i} \geq \lambda^{-} \quad \forall i \tag{34–38}$$
$$x^{+} \geq \hat{x}^{-}_{from\ \mathrm{submodel\ 1}}$$
$$0 \leq \lambda^{-} \leq 1$$
$$x^{+} \geq 0$$

Sub-model 1 is formulated to obtain the upper bound of a maximized minimum goal fulfillment level ($\hat{\lambda}^{+}$) and the corresponding optimal value of the decision variable ($\hat{x}^{-}$). The left-hand side (LHS) of constraint (31) is written considering the maximum possible values of the LHS of Eq. (27). The maximum possible value of the LHS of Eq. (27) occurs with the numerator taking the highest value and the denominator, the lowest. Using the same argument as in Sub-model 1, Sub-model 2 [(34)−(38)] is formulated to obtain the lower bound of the maximized minimum goal fulfillment level ($\hat{\lambda}^{-}$) and corresponding optimal value of decision variable ($\hat{x}^{+}$). The LHS of constraint (35) is written considering the minimum possible values of Eq. (27). The minimum possible value of the left-hand side of Eq. (27) occurs with the numerator taking the lowest value and the denominator, the highest. To ensure that the optimal upper bound of the decision variable $\hat{x}^{+}$, obtained from Sub-model 2 is at least equal to the optimal lower bound of the decision variable $\hat{x}^{-}$, obtained from the Sub-model 1, an interactive constraint (36) (Huang et al., 1995) is added. When the problem is complex and many decision variables with functional relationships are present, a direct comparison of the dominance of x^+ or x^-, i.e., whether x^+ or x^- corresponds to maximized value of λ^+ or λ^-, is impossible to know prior to solving the models. The appropriateness of submodel formulation on finding out suitable deterministic equivalent of the grey fuzzy optimization model depends on the values of interval-valued membership parameters and the consequent intersection of the grey fuzzy goals (Karmakar and Mujumdar, 2005b). For a given set of interval-valued membership parameters, if a particular formulation represents an appropriate deterministic equivalent, other alternative formulations do not. The appropriate deterministic equivalent of a grey optimization model should give the lowest value of grey degree of $\lambda^\pm$.

The solution approach for the fuzzy optimization problem using the max−min operator (Zimmermann, 1978) may not result in a unique solution (Dubois and Fortemps, 1999; Lai and Hwang, 1992; Lin, 2004). To impart flexibility in decision making, the multiple solutions of the fuzzy optimization model may be obtained as a parametric equation or equations that represent a subspace. Determination of such a subspace in a fuzzy optimization problem is itself a potential research area (Lai and Hwang, 1992; Li, 1990; Lin, 2004). It is also observed that as the number of objectives and decision variables increases in the fuzzy optimization model, the possibility of existence of multiple solution increases. When the deterministic equivalents of the grey fuzzy optimization model lead to fuzzy optimization models with a max−min operator, therefore, attention must be given to multiple solutions. Solutions from the grey fuzzy optimization model enhance the flexibility and applicability in decision making, as the decision maker gets a range of optimal solutions, $[\hat{x}^-, \hat{x}^+]$. The width of the interval-valued solutions thus plays an important role in the grey fuzzy optimization model. The grey fuzzy multi-objective optimization technique discussed in the next section maximizes the width of the interval-valued decision variables, ($x^+ - x^-$), (Huang and Loucks, 2000; Karmakar and Mujumdar, 2005a) in a multi-objective framework. Similar to the grey fuzzy optimization model, the upper and lower bounds of the goal fulfillment level (i.e., λ^+ and λ^-) are maximized in the grey fuzzy multi-objective optimization technique, but additionally the width of the degree of goal fulfillment level, ($\lambda^+ - \lambda^-$) is also minimized, thus reducing the system uncertainty.

5. GREY FUZZY MULTI-OBJECTIVE OPTIMIZATION

The grey fuzzy optimization model given in Eqs. (26)−(29) forms the basis of the grey fuzzy multi-objective optimization technique. The inequality constraint (27) addresses the grey fuzzy management goals in the optimization model. The constraint set (27) defines the order relations (e.g., the relations "greater than or equal to" or "less than or equal to") containing interval grey numbers on both sides. Determination of meaningful ranking between two partially or fully overlapping intervals in the order relations is a potential research area (e.g., Ishibuchi and Tanaka, 1990; Moore, 1979; Sengupta et al., 2001). Recently, Sengupta et al., (2001) proposed a satisfactory deterministic equivalent form of inequality constraints containing interval grey numbers by using the acceptability index (***A***). The acceptability index (***A***) is defined as the grade of acceptability of the premise that the "first interval grey number ($a^{\pm}$) is inferior to the second ($b^{\pm}$)," denoted as $a^{\pm}$ (<) $b^{\pm}$. Here, the term "inferior to" ("superior to") is analogous to "less than" ("greater than"). The acceptability index (***A***) is expressed as (Sengupta et al., 2001)

$$\boldsymbol{A}\,[\,a^{\pm}\ (<)\ b^{\pm}\,] = [m(b^{\pm}) - m(a^{\pm})]/[w(b^{\pm}) + w(a^{\pm})] \tag{39}$$

where $[w(b^{\pm}) + w(a^{\pm})] \neq 0$; $w(a^{\pm})$ is the half-width of $a^{\pm} = ½\,(a^{+} - a^{-})$; $m(a^{\pm})$ is the mean of $a^{\pm} = ½\,(a^{-} + a^{+})$. Notations are similarly defined for the interval grey number $b^{\pm}$. The grade of acceptability of $a^{\pm}$ (<) $b^{\pm}$ may be classified and interpreted further on the basis of the comparative position of mean and the half-width of interval $b^{\pm}$ with respect to those of interval $a^{\pm}$. Let us consider an interval inequality relation $a^{\pm}\,y \geq b^{\pm}$, where y is a deterministic variable. A satisfactory deterministic equivalent form of interval inequality relation $a^{\pm}\,y \geq b^{\pm}$, is proposed as (Sengupta et al., 2001):

$$a^{\pm}y \geq b^{\pm} \Rightarrow \{\,a^{-}\mathrm{y} \geq b^{-}\ \text{and}\ \boldsymbol{A}\,[\,a^{\pm}\mathrm{y}\ (<)\ b^{\pm}\,] \leq \alpha \in [0, 1]\} \tag{40}$$

where α is interpreted as an optimistic threshold fixed by the decision maker. Similarly, a satisfactory deterministic equivalent form of interval inequality relation $a^{\pm}y \leq b^{\pm}$ is proposed as (Sengupta et al., 2001):

$$a^{\pm}y \leq b^{\pm} \Rightarrow \{\,a^{+}y \leq b^{+}\ \text{and}\ \boldsymbol{A}\,[\,a^{\pm}y (>) b^{\pm}\,] \leq \alpha \in [0, 1]\} \tag{41}$$

where the symbol (>) indicates "superior to," which is analogous to "greater than." The deterministic equivalent of the grey fuzzy optimization model given in Eqs. (26)−(29) is formulated using the expression (40). By using the attributes mean, width, and acceptability index of the interval grey numbers, the grey fuzzy optimization model is reduced to a deterministic multi-objective optimization model, as follows:

$$\begin{aligned}&\text{Maximize } \lambda^{+}\\ &\text{Maximize } \lambda^{-}\\ &\text{Minimize } [(\lambda^{+}-\lambda^{-})/(\lambda^{+}+\lambda^{-})]\\ &\text{subject to}\\ &\mu_{\mathrm{i}}^{-}(x^{\pm}) = [\{d_i'^{-} - (B_i x)^{+}\}/(d_i'^{+} - d_i^{-})] \geq \lambda^{-} \quad \forall\ i\end{aligned} \qquad (42\text{–}45)$$

$$\boldsymbol{A}\,[\{d_i'^{\pm}-(B_i\,x)^{\pm}\}/(d_i'^{\pm}-d_i^{\pm})\ (<)\,\lambda^{\pm}] \leq \alpha_i \ \in [0, 1] \ \ \forall \qquad (46)$$

$$\begin{aligned}&\mathrm{x}^{-} \leq \mathrm{x}^{+}\\ &\mathrm{x}^{-} \geq 0,\ \mathrm{x}^{+} \geq 0\\ &0 \leq \lambda^{-} \leq 1,\ 0 \leq \lambda^{+} \leq 1\\ &\lambda^{-} \leq \lambda^{+}\end{aligned} \qquad (47\text{–}50)$$

The constraints (45)−(46) together define the deterministic equivalent of the constraint (27). The acceptability index in constraint (46) compares the interval grey numbers in the inequality constraints (27). In constraints (46), α_{i} is the optimistic threshold for the *i*-th constraint fixed by the decision maker. In this model, the grey fuzzy management goal as expressed by constraint (27), is represented by linear imprecise membership functions [i.e., substituting $\gamma_i = 1$ in constraint (27)] as the Eq. (40) with acceptability index for ranking the interval grey numbers in the inequality constraints is applicable only for linear programming problems (Sengupta et al., 2001). The expression (39) of the acceptability index may be substituted in the constraint (46), to obtain a simplified form with algebraic operations on the interval grey numbers (Liu and Lin, 1998; Moore, 1979). The objectives (42) and (43) maximize the upper and lower bound of the goal fulfillment level ($\lambda^{\pm}$), respectively, which ensure the maximum possibility of fulfillment of the grey fuzzy goal. The objective (44) minimizes the width of the goal fulfillment level [$\lambda_w = (\lambda^+ - \lambda^-)$] with maximization of the denominator, $(\lambda^+ + \lambda^-)$, to be consistent with the first two objectives, (42) and (43). This objective is included as the reduction of the width of the goal fulfillment level implies reduction in system uncertainties and an increase in effectiveness of the grey model (Huang et al., 1995). Similarly, a higher flexibility (i.e., higher width of the interval) of the decision variables ($x^\pm$) is always desirable, as it allows a wider choice to the decision-makers. The objectives (42) – (44) do not address the maximization of the width of decision variables [i.e., ($x^+ - x^-$)]. The width of the decision variables may be maximized along with the objectives (42)–(44) while solving the multi-objective optimization model [(42)–(50)] using the fuzzy multi-objective optimization technique (Sakawa, 1984) or the fuzzy goal programming technique (Pal and Moitra, 2003; Sakawa et al., 1987). The procedure of solution is discussed through an application in the next section.

5.1 An Application in Environmental Engineering

A number of successful applications of grey systems theory have been found in many areas of human endeavors, including agriculture, transportation, hydrology, environment, economics, water resources systems, and control theory. Table 3 shows some recent applications of grey systems theory in the field of environmental and water resources engineering. An application of grey fuzzy multi-objective optimization technique is demonstrated with a waste load allocation problem here.

Waste load allocation (WLA) in a stream refers to the determination of required treatment levels of pollutants (fractional removal levels) [e.g., biochemical oxygen demand (BOD) loading, toxic pollutant concentration, etc.] at a set of point sources of pollution to ensure that water quality is maintained at desired levels throughout the stream. A common practice of the pollution control agency (PCA) to ensure an acceptable water quality condition is to check the water quality at a finite number of locations in the river. These locations are called water quality checkpoints. A WLA model for decision making in water quality control in a river system, in general, integrates a water quality simulation model, measuring the influence of a pollutant on a water quality indicator [e.g., dissolved oxygen (DO) deficit, hardness, nitrate-nitrogen concentration, etc.] at a downstream location with an optimization model to provide best compromise solutions acceptable to both PCA and dischargers (e.g., municipal and industrial

Table 3. Applications of Grey Systems Theory in Water Resources and Environmental Engineering

Application	Literature	Case study	Parameters considered as interval grey numbers
Municipal solid waste management and planning	Chen and Huang (2001), Huang et al. (1992, 1993), Zou et al. (2000)	Hypothetical study area	Existing landfill capacity, treatment capacity, generated residues after treatment, revenue from waste-to-energy facility.
Water quantity & quality management	Huang et al. (1996), Huang and Loucks (2000)	1. Fujian province of China, 2. Hypothetical study area	Water quantity: crop water requirement, municipal water requirement, cost for obtaining-transporting-delivering-allocating water, cost of manure collection, cost of fertilizer application, average returns from livestocks, etc.; Water quality: amount of manure generated by humans, livestocks, amount of manure applied to soil, population in study area, number of livestock, nitrogen volatilization and denitrification rates, area under the crops, pollutant losses, etc.
Rainfall forecasting	Yu et al. (2000)	San-Hsia and Heng-Chi subcatchments in Tahan creek, Taiwan	Areal mean rainfall.
Reservoir operation	Chang et al. (2002)	Shiman reservoir in Taiwan	Storage of the upper and lower curves in rule curves at *i*-th stage, inflows, supply for irrigation, municipal and industrial purposes.
Water quality control problems (rivers and lakes)	Chang et al. (1997), Wu et al. (1997), Karmakar and Mujumdar (2005a, 2005b)	1. Tseng-Wen river basin in south Taiwan, 2. Lake Erhai in southwestern China	Degree of aspiration levels, BOD loading, construction and average operating cost of treatment plants, removal efficiency of BOD, deoxygenation and reaeration coefficients.
Coastal waste water treatment	Chang and Wang (1995)	Guishuic waste water treatment project in Taiwan	Concentrations of conservative pollutants, waste water flow rates, initial dilution, length of diffusers.

dischargers). A number of WLA models have been developed in the past for optimal allocation of assimilative capacity of a river system considering uncertainties due to randomness in input variables (e.g., stream flow, effluent flow, temperature, reaction rates, etc.) and imprecision in management goals (e.g., goals of PCA and dischargers), the latter being addressed using the fuzzy sets theory. Imprecision in management goals is usually modeled using fuzzy membership functions, specifying the desirable maximum permissible levels of the goals by prespecified membership parameters. Choice of appropriate values of membership parameters is an important issue in any fuzzy optimization model, as these are highly subjective. In a water quality control problem, such subjectivity in choice of parameters results in an uncertainty in the membership parameters and leads to a second level of fuzziness in the model, with the membership functions themselves being imprecisely stated. Moreover, in practical situations, for the same water quality indicator, different water quality standards are used for different uses, which results in an uncertainty in the membership parameters of the goals of PCA.

Two sets of conflicting goals associated with the river water quality management are generally considered in a waste load allocation problem. The PCA specifies the desirable concentration level ($c^D{}_{jl}$) and maximum permissible concentration level ($c^H{}_{jl}$) of the water quality indicator j at the water quality checkpoint l ($c^D{}_{jl} \leq c^H{}_{jl}$). The goal of the PCA ($E^{\pm}_{jl}$) is to make the concentration level (c_{jl}) of water quality indicator j at the checkpoint l as close as possible to the desirable level, $c^D{}_{jl}$, so that the water quality at the checkpoint l is enhanced with respect to the water quality indicator j, for all j and l. This goal is represented by a membership function. For example, if the DO-deficit is the water quality indicator, a non-increasing membership function suitably reflects the goals of the PCA with respect to DO-deficit at a checkpoint. The uncertainty associated with membership parameters ($c^D{}_{jl}$ and $c^H{}_{jl}$) is addressed using interval grey numbers, and the membership parameters are expressed as $c^{D\pm}{}_{jl}$ and $c^{H\pm}{}_{jl}$. Using nonincreasing imprecise membership functions, the grey fuzzy goals of PCA are expressed as

$$\mu^{\pm}_{E^{\pm}_{jl}}(c^{\pm}_{jl}) = \begin{cases} 1 & c^{+}_{jl} < c^{D-}_{jl} \\ [(c^{H\pm}_{jl} - c^{\pm}_{jl})/(c^{H\pm}_{jl} - c^{D\pm}_{jl})]^{\gamma jl} & c^{D\pm}_{jl} \leq c^{\pm}_{jl} \leq c^{H\pm}_{jl} \\ 0 & c^{-}_{jl} > c^{H+}_{jl} \end{cases} \tag{51}$$

The exponent γ_{jl} is a nonzero positive real number. Assignment of numerical value to this exponent is subject to the desired shape of the membership functions. A value of $\gamma_{jl} = 1$ leads to a linear imprecise membership function. The grey fuzzy goals of the dischargers are similarly expressed as:

$$\mu^{\pm}_{F^{\pm}_{jmn}}(x^{\pm}_{jmn}) = \begin{cases} 1 & x^{+}_{jmn} < x^{L-}_{mn} \\ [(x^{M\pm}_{mn} - x^{\pm}_{jmn})/(x^{M\pm}_{mn} - x^{L\pm}_{mn})]^{\beta_{jmn}} & x^{L\pm}_{mn} \le x^{\pm}_{jmn} \le x^{M}_{mn} \\ 0 & x^{-}_{jmn} > x^{M+}_{mn} \end{cases} \tag{52}$$

where the aspiration level and the maximum acceptable level of fractional removal of the pollutant *n* at discharger *m* are represented as $x^{L\pm}_{mn}$ and $x^{M\pm}_{mn}$, respectively ($x^{L\pm}_{mn} \le x^{M\pm}_{mn}$). Similar to the exponent γ_{jl} in Eq. (51), β_{jmn} is a nonzero positive real number. The goal of the dischargers ($F^{\pm}_{jmn}$) is to make the fractional removal level ($x^{\pm}_{jmn}$) as close as possible to $x^{L\pm}_{mn}$, to minimize the waste treatment cost for pollutant *n*. These two sets of conflicting grey fuzzy goals are incorporated in the optimization model using the grey fuzzy decision concept. Using the concept of grey fuzzy optimization discussed in Section 4, the grey fuzzy waste load allocation model (GFWLAM) is written as (Karmakar and Mujumdar, 2005b):

$$\begin{aligned} &\text{Maximize} \quad \lambda^{\pm} \\ &\text{subject to} \\ &\mu^{\pm}_{E^{\pm}_{jl}}(c^{\pm}_{jl}) = [(c^{H\pm}_{jl} - c^{\pm}_{jl})/(c^{H\pm}_{jl} - c^{D\pm}_{jl})]^{\gamma_{jl}} \ge \lambda^{\pm} && \forall\ j,l \\ &\mu^{\pm}_{F^{\pm}_{jmn}}(x^{\pm}_{jmn}) = [(x^{M\pm}_{mn} - x^{\pm}_{jmn})/(x^{M\pm}_{mn} - x^{L\pm}_{mn})]^{\beta_{jmn}} \ge \lambda^{\pm} && \forall\ j,m,n \\ &c^{D\pm}_{jl} \le c^{\pm}_{jl} \le c^{H\pm}_{jl} && \forall\ j,l \\ &x^{L\pm}_{mn} \le x^{\pm}_{jmn} \le x^{M\pm}_{mn} && \forall\ j,m,n \\ &0 \le \lambda^{\pm} \le 1 \end{aligned} \tag{53–58}$$

The constraints (54) and (55) are constructed from imprecise membership functions for the grey fuzzy goals of PCA and dischargers, respectively. The crisp constraints (56) and (57) are based on the water quality requirements specified by the PCA and possible fractional removal levels, respectively. Constraint (58) represents the bounds on the parameter $\lambda^{\pm}$. In the expression for goals of PCA [constraint (54)], the concentration level $c^{\pm}_{jl}$, of water quality indicator j at checkpoint l, may be mathematically expressed as:

$$c^{\pm}_{jl} = f(x^{\pm}_{jmn}) \tag{59}$$

where the transfer function f indicates the aggregated effect of all pollutants and dischargers (located upstream of checkpoint l) on the water quality indicator j. The transfer function can be evaluated using appropriate mathematical models that determine spatial distribution of the water quality indicator due to pollutant discharge into the river system from point sources (Sasikumar and Mujumdar, 1998; Mujumdar and Sasikumar, 2002). The fractional removal levels ($x^{\pm}_{\text{jmn}}$) and the goal fulfillment level ($\lambda^{\pm}$) are the decision variables in this model. The grey fuzzy inequality constraints (54) and (55) addressing the goals of the PCA and dischargers are the order relations containing interval grey numbers on both sides. A satisfactory deterministic equivalent of these constraints can be obtained using the concept of acceptability index (***A***) as defined in Eq. (40). The deterministic equivalent of the grey fuzzy optimization model (53)−(58) can be formulated using the methodology of formulating the multi-objective optimization model as presented in (42)−(50) and expressed as

$$\begin{aligned}
&\text{Maximize } \lambda^{+}\\
&\text{Maximize } \lambda^{-}\\
&\text{Minimize } [(\lambda^{+} - \lambda^{-})/(\lambda^{+} + \lambda^{-})]\\
&\text{subject to}\\
&\mu^{-}_{\text{E}^{\pm}_{\text{jl}}}(c^{\pm}_{jl}) = [(c^{H-}_{jl} - c^{+}_{jl})/(c^{H+}_{jl} - c^{D-}_{jl})] \ge \lambda^{-} \quad \forall\ j,l\\
&\mu^{-}_{\text{F}^{\pm}_{\text{jmn}}}(x^{\pm}_{jmn}) = [(x^{M-}_{mn} - x^{+}_{jmn})/(x^{M+}_{mn} - x^{L-}_{mn})] \ge \lambda^{-} \quad \forall\ j,\ m,\ n
\end{aligned} \tag{60–64}$$

$$A\,[(c^{H\pm}_{jl} - c^{\pm}_{jl})/(c^{H\pm}_{jl} - c^{D\pm}_{jl})\,(<)\ \lambda^{\pm}] \le \alpha_1 \in [0, 1] \qquad \forall\, j, l \tag{65}$$

$$A\,[(x^{M\pm}_{jmn} - x^{\pm}_{jmn})/(x^{M\pm}_{jmn} - x^{L\pm}_{jmn})\,(<)\ \lambda^{\pm}] \le \alpha_2 \in [0, 1] \quad \forall\, j, m, n \tag{66}$$

$$
\begin{aligned}
& c_{jl}^{D-} \le c_{jl}^{+} \le c_{jl}^{H+}; \quad c_{jl}^{D-} \le c_{jl}^{-} \le c_{jl}^{H+} && \forall\, j, l \\
& x_{mn}^{L-} \le x_{jmn}^{+} \le x_{mn}^{M+}; \quad x_{mn}^{L-} \le x_{jmn}^{-} \le x_{mn}^{M+} && \forall\, j, m, n \\
& c_{jl}^{-} \le c_{jl}^{+} && \forall\, j, l \\
& x_{jmn}^{-} \le x_{jmn}^{+} && \forall\, j, m, n \\
& 0 \le \lambda^{+} \le 1; \quad 0 \le \lambda^{-} \le 1 \\
& \lambda^{-} \le \lambda^{+}
\end{aligned}
\qquad (67\text{–}72)
$$

The constraints (63)−(66) are the deterministic equivalent of the constraints (54)−(55) using the acceptability index. The goals of the PCA and dischargers are represented by linear imprecise membership functions (i.e., γ_{jl}, $\beta_{jmn} = 1$). The objective (62) minimizes the system uncertainty by minimizing $[(\lambda^{+}-\lambda^{-})/(\lambda^{+}+\lambda^{-})]$. In the multi-objective optimization model (60)–(72), the three objectives are optimized individually in three separate sub-problems along with the contraints (63)−(72) to obtain the maximum and minimum possible values of λ^{+}, λ^{-} and $[(\lambda^{+}-\lambda^{-})/(\lambda^{+}+\lambda^{-})]$ [i.e., ideal points and worst possible values of the fuzzy multi-objective optimization technique (Sakawa, 1984)], respectively. As discussed earlier, another objective of the river water quality management is to permit more flexibility (i.e., more width of the interval) in the optimal fractional removal level ($\hat{x}_{jmn}^{\pm}$). Thus, maximization of the grey degree of $x_{jmn}^{\pm}$ is considered as another objective along with objectives (60) to (62). The maximum and minimum values of the grey degree of $x^{\pm}_{jmn}$ are determined from the three sub-problems. All the objectives are quantified by using appropriate membership functions according to the fuzzy multi-objective optimization technique (Sakawa, 1984). The fuzzy decision concept with a "minimum" operator is applied to aggregate the membership functions of the objectives (60)−(62) along with other objectives for minimizing $Gd\,(x^{\pm}_{jmn})$ for the dischargers. The solution algorithm for the problem (60)–(72) is as follows: (1) Solve three sub-problems, each formulated with one objective (λ^{+}, λ^{-} and $[(\lambda^{+}-\lambda^{-})/(\lambda^{+}+\lambda^{-})]$) and all constraints. (2) From the three sets of solutions, obtain the best and worst values of λ^{+}, λ^{-}, $[(\lambda^{+}-\lambda^{-})/(\lambda^{+}+\lambda^{-})]$ and $Gd(x^{\pm}_{jmn})$. (3) Define membership functions for λ^{+} (non-decreasing), λ^{-} (non decreasing), $[(\lambda^{+}-\lambda^{-})/(\lambda^{+}+\lambda^{-})]$ (non increasing) and $Gd(x^{\pm}_{jmn})$ (non decreasing) with their best and worst values. (4) Maximize the minimum membership of the objectives using the fuzzy decision concept with the max–min approach. This gives the solution for the grey fuzzy multiobjective optimnization problem, Eqs. (60)−(72). Application of the grey fuzzy multi-objective optimization model [Eqs. (60)−(72)] for water quality management is demonstrated on a hypothetical river system shown in Figure 5.

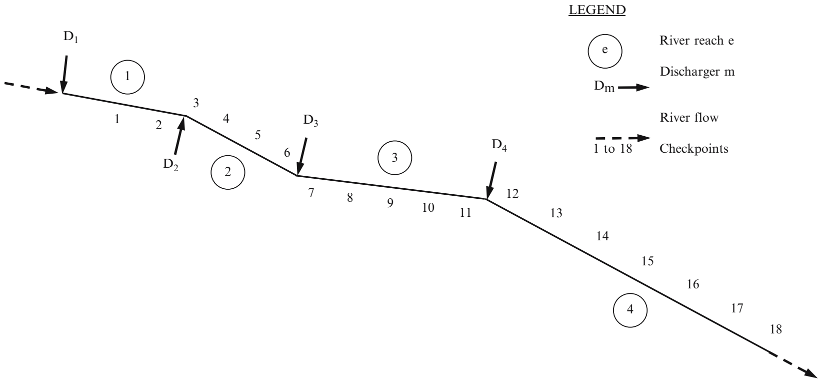

Figure 5. Hypothetical river system

In this application, the water quality indicator of interest is the DO-deficit at 18 checkpoints in the river system due to the point sources of BOD from four dischargers. The saturation DO concentration is taken as 10 mg/L for all the reaches. A deterministic value of river flow of 7 Mcum/day is considered. The notations of variables are simplified by retaining only the suffixes l (checkpoints) and *m* (dischargers) in the model (60)−(72) dropping the suffixes *j* and *n* as there is only one water quality indicator (DO deficit) and only one pollutant (BOD). Details of the effluent flow and imprecise membership functions are given in Tables 4 and 5, respectively (* Data from Mujumdar and Sasikumar, 2002).

Table 4. Effluent Flow Data*

Discharger	Effluent flow rate (10^4 m^3/day)	BOD (mg/L)	DO (mg/L)
1	2.134	1250	1.230
2	6.321	1415	2.400
3	7.554	1040	1.700
4	5.180	935	2.160

Table 5. Details of Imprecise Membership Functions*

River reach	Check-points	c^D_l (mg/L)		c^H_l (mg/L)		x^L_m		x^M_m	
		–	+	–	+	–	+	–	+
1	1−2	(0.00) 0.00	0.00	(3.00 2.70	3.20	(0.30) 0.25	0.35	(0.85) 0.80	0.90
2	3−6	(0.10) 0.00	0.10	(3.00) 2.70	3.20	(0.30) 0.25	0.35	(0.85) 0.80	0.90
3	7−11	(0.20) 0.17	0.22	(3.50) 3.30	3.70	(0.35) 0.30	0.40	(0.85) 0.80	0.90
4	12−18	(0.20) 0.17	0.22	(3.50) 3.30	3.70	(0.35) 0.30	0.40	(0.85) 0.80	0.90

( ): Deterministic values of membership parameters, "–" : Lower bound, "+" : Upper bound, "*": Data from Karmakar and Mujumdar (2005a)

In constraints (65) and (66), α_1 and α_2 are optimistic thresholds, which are set to zero in the current application, and thus, a conservative optimal solution is obtained, implying a stringent restriction on water pollution. The decision maker selects the values of α_1 and α_2 equal to zero when the water quality management issues in the river system are too critical and important; otherwise some optimistic strategy can be considered by choosing values of α_1 and α_2 close to unity. For most water quality indicators, a high level of fractional removal of pollutants (e.g., BOD loading, toxic pollutant concentration, etc.) results in a low level of water quality indicator (e.g., DO-deficit, nitrate-nitrogen concentration, etc.). The lower bound of water quality indicator (c^-_l) is therefore expressed in terms of the upper bound of fractional removal level (x^+_m) and similarly, c^+_l is expressed in terms of x^-_m, using the one-dimensional Streeter−Phelps model for a BOD–DO relationship in a stream (Streeter and Phelps, 1925). Further, using the recursive relationships given by Fugiwara et al., 1987; 1988), the DO-deficit is written as a linear function of the fractional removal levels. This results in $x^{\pm}_{jmn}$ as the only decision variables in the optimization model (60)−(72). Table 6 shows the expressions of the DO-deficit at the 18 checkpoints in terms of fractional removal levels of BOD waste load by dischargers, situated upstream of the particular checkpoint. For example, the DO-deficit at checkpoint 15 is expressed as follows using the data given in Table 6:

$$c_{15} = 0.309 - 0.201\ \ x_1 - 0.079\ \ x_2 - 0.024\ \ x_3 - 0.002x_4 \tag{73}$$

Table 6. DO-Deficit

River reach	Check-points	Constant terms	(−1) × Coefficients of fractional removal levels			
			x_1	x_2	x_3	x_4
1	1	0.1142	0.0893	—	—	—
	2	0.1935	0.1702	—	—	—
2	3	0.2595	0.1687	—	—	—
	4	0.6198	0.2409	0.2942	—	—
	5	0.9472	0.3062	0.5618	—	—
	6	1.2433	0.3651	0.8042	—	—
3	7	1.3230	0.3628	0.7992	—	—
	8	1.8327	0.4149	1.0150	0.2524	—
	9	2.2948	0.4620	1.2095	0.4828	—
	10	2.7140	0.5041	1.3862	0.6923	—
	11	3.0919	0.5418	1.5453	0.8821	—
4	12	3.1175	0.5361	1.5289	0.8734	—
	13	3.6006	0.5691	1.6690	1.0412	0.1538
	14	4.0345	0.5980	1.7928	1.1937	0.2935
	15	4.4259	0.6238	1.9039	1.3309	0.4211
	16	4.7756	0.6462	2.0023	1.4538	0.5366
	17	5.0877	0.6656	2.0894	1.5635	0.6413
	18	5.3635	0.6823	2.1652	1.6611	0.7354

Substituting the values of membership parameters and the expressions of DO-deficit in terms of BOD removal levels from Tables 5/6, respectively, in the grey fuzzy multi-objective optimization model (60)–(72) and solving the resulting linear programming problem, optimal interval-valued fractional removal levels of BOD are determined as presented in Table 7. In Table 7, columns 2–4 show the results obtained from Sub-problems 1–3, i.e., maximization of λ^+, maximization of λ^- and minimization of $[(\lambda^+-\lambda^-)/(\lambda^+ +\lambda^-)]$, respectively. The minimum and maximum values of λ^+, λ , $[(\lambda^+-\lambda^-) / (\lambda^++\lambda^-)]$ and $Gd(x^\pm_1)$, … , $Gd(x^\pm_4)$ are taken from the columns 2–4; rows 6, 5, 8, and 9–12, respectively. For example, columns 2–4, row 5, show the values of λ^- obtained from Sub-problem 1–3, respectively. The maximum value of λ^- (i.e., 0.3121) is obtained from Sub-problem 2, and the minimum value (i.e., 0.0006) is obtained from Sub-problem 1. The requirements of all the objectives are quantified by defining linear membership functions with the minimum and maximum values of the objective functions as membership parameters. Column 5, rows 1−6, show the optimal fractional removal levels of the pollutants by different dischargers ($\hat{x}^\pm$) and corresponding $\hat{\lambda}^\pm$ values.

Table 7. BOD

Sl. No.	Solution (1)	Sub-problem 1 (Max.λ^+) (2)	Sub-problem 2 (Max.λ) (3)	Sub-problem 3 [Min.$(\lambda^+ - \lambda^-)/(\lambda^+ + \lambda^-)$] (4)	Multi-objective GFWLAM (5)	Deterministic model (6)	GFWLAM (7)
1	X_1	[0.4845, 0.7751]	[0.5955, 0.5964]	[0.6060, 0.6531]	[0.5268, 0.6652]	[0.6150, 0.6150]	[0.5970, 0.6410]
2	X_2	[0.4682, 0.7987]	[0.5967, 0.5970]	[0.4301, 0.6578]	[0.5302, 0.6656]	[0.6150, 0.6150]	[0.5970, 0.6410]
3	X_3	[0.5190, 0.7951]	[0.6124, 0.6126]	[0.5517, 0.5679]	[0.5367, 0.6757]	[0.6360, 0.6360]	[0.6120, 0.6700]
4	X_4	[0.5172, 0.7970]	[0.6121, 0.6127]	[0.6324, 0.6517]	[0.5357, 0.6747]	[0.6360, 0.6360]	[0.6120, 0.6700]
5	λ^-	0.0006	0.3121	0.1052	0.2066	0.4277	0.3126
6	λ^+	0.9592	0.3618	0.1052	0.5064	0.4277	0.5745
7	Gd $(\lambda^{\pm})$	—	—	—	0.8411	0.0000	0.5903
8	$(\lambda^+ - \lambda^-)/(\lambda^+ + \lambda^-)$	0.9987	0.0737	0.0000	—	—	—
9	$Gd(x^{\pm}_1)$	0.4615	0.0015	0.0748	0.2322	0.0000	0.0711
10	$Gd\ (x^{\pm}_2)$	0.5217	0.0004	0.4186	0.2265	0.0000	0.0711
11	$Gd\ (x^{\pm}_3)$	0.4202	0.0003	0.0288	0.2293	0.0000	0.0905
12	$Gd\ (x^{\pm}_4)$	0.4259	0.0010	0.0300	0.2297	0.0000	0.0905
13	Avg. $Gd\ (x^{\pm})$	—	—	—	0.2294	0.0000	0.0812

To evaluate the quality of input or output uncertain information, a measure of "Grey degree" [Eq. (4)] is used. As the grey degree of the optimal value of the objective function decreases, the effectiveness of the grey model increases with decreasing system uncertainties. Substituting the deterministic values of membership parameters given in Table 5, in the grey fuzzy optimization model (53)−(58), the optimal fractional removal levels of BOD are determined as presented in column 6 of Table 7, for which average value of the grey degree of input parameters is zero. In column 7 of Table 7, the solutions obtained from GFWLAM based on the grey fuzzy optimization technique (Section 4) are presented. For this solution, values of all input parameters are considered the same as those for the multi-objective GFWLAM. Comparing the results shown in column 5 and column 7 (rows 1–4) it may be concluded that the widths of optimal fractional removal levels of BOD ($\hat{x}^{\pm}$) for multi-objective GFWLAM are more than those of GFWLAM because of the inclusion of the objective of maximization of grey degrees of fractional removal levels. The same observation can also be made by comparing the $Gd(\hat{x}^{\pm})$ values shown in rows 9–12 of columns 5 and 7. The value of $Gd(\hat{\lambda}^{\pm})$ is, however, more than the value obtained from GFWLAM, which indicates more uncertainty in the system compared with that resulting from the GFWLAM solution. The result obtained from multi-objective GFWLAM is more useful to the decision makers as it gives a wider range in the interval-valued optimal fractional removal levels of the pollutants than GFWLAM, although at the cost of increasing uncertainty, in this particular application.

The current application of grey fuzzy multi-objective optimization technique on waste load allocation demonstrates the modeling aspects of uncertain membership functions for different management goals and shows the usefulness of solutions with a simplified hypothetical river system. Although the solutions obtained from the grey fuzzy multi-objective optimization model (i.e., multi-objective GFWLAM) provide more flexibility than those obtained from the grey fuzzy optimization model (i.e., GFWLAM), the application of multi-objective GFWLAM is limited to grey fuzzy goals expressed by linear imprecise membership functions, whereas GFWLAM has the capability to solve the grey fuzzy optimization model with monotonic, nonlinear, imprecise membership functions with γ_{jl} and $\beta_{jmn} \neq 1$, in Eqs. (51) and (52).

6. CONCLUSION

An overview of grey fuzzy optimization techniques are presented in this chapter. The concept of fuzzy decision is extended to grey fuzzy decision by considering the uncertainty in membership parameters using grey systems theory. A brief description of grey systems theory is presented as a prerequisite for understanding the grey fuzzy optimization technique. The grey fuzzy optimization model is further enhanced to multi-objective framework to maximize the width of the optimal interval-valued decision variables providing latitude in decision making and to minimize the width of the goal fulfillment level for reducing the system uncertainty. The concept of acceptability index for order relation between two partially or fully overlapping intervals is used to get a deterministic equivalent of the grey fuzzy optimization model. Although the solutions obtained from the multi-objective optimization model provide more flexibility in decision making than those obtained by the grey fuzzy optimization model, the application of the multi-objective optimization model is limited to grey fuzzy goals expressed by linear imprecise membership functions, whereas the grey fuzzy optimization model has the capability to solve the model with monotonic, nonlinear, imprecise membership functions also. The application of the models is demonstrated with the problem of waste load allocation, in the field of environmental engineering.

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FUZZY MULTI-OBJECTIVE DECISION-MAKING MODELS AND APPROACHES

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Abstract: Multi-objective linear programming (MOLP) techniques are widely used to model many organizational decision problems. Referring to the imprecision inherent in human judgments, uncertainty may be incorporated in some parameters of an established MOLP model that is also called a fuzzy MOLP (FMOLP) problem. This chapter first reviews the development of fuzzy multi-objective decision-making (FMODM) models and approaches and then proposes an effective way for an optimal solution in the FMOLP problem. By introducing an adjustable satisfactory degree α, a new concept of FMOLP and a solution transformation theorem are given in this chapter. This chapter thus develops an interactive fuzzy goal multi-objective decision-making method, which provides an interactive fashion with decision makers during their solution process and allows decision makers to give their fuzzy goals in any form of membership function. An illustrative example shows the details of the proposed method.

Key words: Fuzzy programming, multi-objective linear programming, interactive multi-objective decision-making method

1. INTRODUCTION

Many organizational decision problems are involved in multiple objectives, called multi-objective decision making (MODM). Most MODM problems can be formulated by multi-objective linear programming (MOLP) models. Referring to the imprecision and insufficient inherent in human judgments, uncertainty may be affected and

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incorporated in some parameters of an MOLP model. Such a model is often called a fuzzy MOLP (FMOLP) model.

Various methods have been proposed from the literature to derive a satisfaction solution of an MOLP problem for decision makers based on their subjective value judgment and preference. Two main types of such methods are goal programming and interactive programming (Hwang and Masud, 1979). In general, there is no unique solution for both MOLP and FMOLP problems. To obtain a satisfactory solution of an FMOLP problem for a particular decision maker involves a lot of interaction to carry out the decision maker's preference for a solution. When both the parameters in the model and the goals given by a decision maker are with uncertainty the interactive solution procedure may become very complex, and therefore, more efficient FMOLP methods are needed.

Many optimization methods and techniques for modeling and solving FMOLP problems have been proposed (Carlsson and Fuller, 1996; Inuiguchi and Ramik, 2000; Lai and Hwang, 1994; Sakawa, 1993a). Fuzzy numbers seem promising to model and solve an FMOLP problem. Many applications have also proved it applicable for dealing with human decision-making problems in most practical situations (Bellmann and Zadeh, 1970; Sakawa, 1993b). Tanaka and Asai (1984) formulated FMOLP problems by using triangular fuzzy numbers to describe the fuzzy parameters in both objective functions and constraints. Lai and Hwang (1992) also modeled FMOLP problems by using triangular fuzzy numbers and solved FMOLP problems by the fuzzy ranking concept as well to handle imprecise constraints. Luhandjula (1987) proposed the concepts of α-possible feasibility and β-possible efficiency based on fuzzy numbers and used the two concepts to solve the FMOLP problem by transferring it into an auxiliary crisp MOLP problem. Furthermore, Slowinski (1990) proposed an interactive method (FLIP) for solving MOLP problems with fuzzy parameters in the objective functions and on the both sides of the constraints. Rommelfanger (1989, 1990) presented a method (FULPAL) for solving (multi-criteria) linear programs, where the right-hand sides as well as the parameters in the constraints and/or the objective functions may be fuzzy. Similarly, Ramik and Rommelfanger (1993, 1996) proposed a unified approach based on the fuzzy inequality relations for the fuzzy mathematical programming problem in which fuzzy parameters may have nonlinear membership functions. In particular, Inuiguchi and Ramik (2000) reviewed some fuzzy linear programming methods and techniques from a practical point of view and introduced the general history and the approaches of fuzzy mathematical programming. In the meantime, goal programming (Charnes and Cooper, 1977) as an effective method has been successfully applied in solving FMOLP problems. Kuwano (1996) applied the concepts of the α-optimal solution and the restricted α-optimal value at the α-optimal solution to establish a goal programming approach for solving FMOLP problems. Sakawa and Nishizaki (2000) pushed the work forward based on their previous results (Sakawa, 1993a; Sakawa and Yano, 1990) by defining two new concepts for FMOLP based on fuzzy goals. One is defined by maximizing the minimal fuzzy goal and the other by maximizing the sum of fuzzy goals. They then developed two computational methods for obtaining the solutions for FMOLP problems. More importantly, Ramik (2000) generated a standard goal programming problem with alternatives and goals being fuzzy sets, and the satisfaction of a goal by a fuzzy objective function is also expressed by a fuzzy relation; he proposed a unifying approach covering several approaches known from the literature.

Although these methods are efficient to solve FMOLP problems, there are two limitations in their current results. One is that only some specialized forms of membership functions such as a triangular form were used to deal with fuzzy parameters and fuzzy goals. This may restrict the use of other forms of membership functions to describe the parameters in modeling an FMOLP problem and to express their goals by decision makers in solving the FMOLP problem. The second limitation is that the values of objective functions, in corresponding to a satisfactory solution of an FLOMP problem, are only described by some crisp values, which is sometimes not appropriate in practice. Since a decision problem is formulated with uncertainty and its solution is received with fuzzy values, it is more reasonable to provide the values of the objective functions with a range in values.

This study, therefore, develops a generalized fuzzy goal fuzzy multi-objective optimization method to assist decision makers to obtain satisfactory solutions for an FMOLP problem. The method can solve the FMOLP problem with whatever the parameters of both objectives and constraints are described in any form of membership functions. The method also allows decision makers to provide their fuzzy goals for the objectives of their decision problems by linguistic terms by any form of membership functions. By introducing an adjustable satisfactory degree α, the obtained values of objective functions, corresponding to a solution, can be described by fuzzy values in which a real number is as a special case. Moreover, the generalized fuzzy goal fuzzy multi-objective decision-making method has the features of interaction with decision makers during a solution process.

The results reported here are our continuing research, and a summary about our previous reports is in (Lu et al., 2007; 2006; Wu et al., 2003; 2004a; 2004b; 2006; Zhang et al., 2002; 2003). This chapter first gives a general FMOLP model where fuzzy parameters of objective functions and constraints are described by membership functions. To solve such an FMOLP problem, an optimal solution concept, a general solution transformation theorem, and a related workable solution transformation theorem are then developed. Based on these theories, an FMOLP problem can be transformed into an MOLP problem. Therefore, an optimal solution of an FMOLP can be obtained through solving an associated MOLP problem. Under this principle, an interactive FMOLP method is presented by 11 steps within two stages. Finally a numeral example illustrates the proposed FMOLP method.

2. FUZZY MULTI-OBJECTIVE DECISION-MAKING MODEL

This section introduces a set of fuzzy multi-objective linear programming models. It then gives the concepts of optimal solutions for such kinds of problems. These models will be applied in the following sections to develop related methods and algorithms to achieve an optimal solution for an FMOLP problem.

2.1 Model and Pareto Optimal Solution for General FMOLP Problems

Consider the following fuzzy multi-objective linear programming (FMOLP) problem:

$$(\text{FMOLP}) \quad \begin{cases} \text{Maximize} & \langle \tilde{c}, x \rangle_F = \left(\sum_{i=1}^{n} \tilde{c}_{1i} x_i , \sum_{i=1}^{n} \tilde{c}_{2i} x_i , \cdots, \sum_{i=1}^{n} \tilde{c}_{ki} x_i \right)^T \\ \text{subject to} & \tilde{A}x \underset{=}{\prec} \tilde{b}, \; x \underset{=}{>} 0, \end{cases} \tag{1}$$

where

$$\tilde{c} = \begin{pmatrix} \tilde{c}_{11} & \tilde{c}_{12} & \cdots & \tilde{c}_{1n} \\ \tilde{c}_{21} & \tilde{c}_{22} & \cdots & \tilde{c}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{c}_{k1} & \tilde{c}_{k2} & \cdots & \tilde{c}_{kn} \end{pmatrix}, \tilde{A} = \begin{pmatrix} \tilde{a}_{11} & \tilde{a}_{12} & \cdots & \tilde{a}_{1n} \\ \tilde{a}_{21} & \tilde{a}_{22} & \cdots & \tilde{a}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{a}_{m1} & \tilde{a}_{m2} & \cdots & \tilde{a}_{mn} \end{pmatrix},$$

$$\tilde{b} = (\tilde{b}_1, \tilde{b}_2, \cdots, \tilde{b}_m)^T \in F \times (R^m),$$

and $\tilde{c}_{sj}$, $\tilde{a}_{ij} \in F^*(R)$, $s = 1, 2, \cdots, k$, $i = 1, 2, \cdots, m$, $j = 1, 2, \cdots, n$.

For the sake of simplicity, we set $\tilde{X} = \{x; \tilde{A}x \preceq \tilde{b}, x \geqq 0\}$ and assume that $\tilde{X}$ is compact. In an FMOLP problem, for each $x \in \tilde{X}$, the value of the objective function $\langle \tilde{c}, x \rangle_F$ is a fuzzy number. Thus, we introduce the following concepts of optimal solutions to FMOLP problems.

DEFINITION 1.

A point $x^* \in R^n$ is said to be a complete optimal solution to the FMOLP problem if it holds that $\langle \tilde{c}, x^* \rangle_F \succeq \langle \tilde{c}, x \rangle_F$ for all $x \in \tilde{X}$.

DEFINITION 2.

A point $x^* \in R^n$ is said to be a Pareto optimal solution to the FMOLP problem if there is no $x \in \tilde{X}$ such that $\langle \tilde{c}, x \rangle_F \succeq \langle \tilde{c}, x^* \rangle_F$ holds.

DEFINITION 3.

A point $x^* \in R^n$ is said to be a weak Pareto optimal solution to the FMOLP problem if there is no $x \in \tilde{X}$ such that $\langle \tilde{c}, x \rangle_F \succ \langle \tilde{c}, x^* \rangle_F$ holds.

The basic ideas to solve the FMOLP problem are (1) to transform it into an associative crisp MOLP problem. (2) As MOLP problems have been well studied, a Pareto optimal solution of the MOLP problem can be obtained. (3) Through setting up the relationship between the solution of the FMOLP and the solution of the associative MOLP, the original FMOLP problem can be solved. Therefore, we first consider the following MOLP problem that is associated with the FMOLP problem shown in (1):

$$\text{(MOLP)} \quad \begin{cases} \text{Maximize} & \left(\langle c_\lambda^L, x \rangle, \langle c_\lambda^R, x \rangle \right)^T \\ \text{subject to} & A_\lambda^L x \leqq b_\lambda^L,\ A_\lambda^R x \leqq b_\lambda^R,\ x \geqq 0, \forall \lambda \in [0,1] \end{cases} \tag{2}$$

where

$$C_\lambda^L = \begin{bmatrix} c_{11\lambda}^L & c_{12\lambda}^L & \cdots & c_{1n\lambda}^L \\ c_{21\lambda}^L & c_{22\lambda}^L & \cdots & c_{2n\lambda}^L \\ \vdots & \vdots & \ddots & \vdots \\ c_{k1\lambda}^L & c_{k2\lambda}^L & \cdots & c_{kn\lambda}^L \end{bmatrix}, \quad C_\lambda^R = \begin{bmatrix} c_{11\lambda}^R & c_{12\lambda}^R & \cdots & c_{1n\lambda}^R \\ c_{21\lambda}^R & c_{22\lambda}^R & \cdots & c_{2n\lambda}^R \\ \vdots & \vdots & \ddots & \vdots \\ c_{k1\lambda}^R & c_{k2\lambda}^R & \cdots & c_{kn\lambda}^R \end{bmatrix}$$

$$A_\lambda^L = \begin{bmatrix} a_{11\lambda}^L & a_{12\lambda}^L & \cdots & a_{1n\lambda}^L \\ a_{21\lambda}^L & a_{22\lambda}^L & \cdots & a_{2n\lambda}^L \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1\lambda}^L & a_{m2\lambda}^L & \cdots & a_{mn\lambda}^L \end{bmatrix}, A_\lambda^R = \begin{bmatrix} a_{11\lambda}^R & a_{12\lambda}^R & \cdots & a_{1n\lambda}^R \\ a_{21\lambda}^R & a_{22\lambda}^R & \cdots & a_{2n\lambda}^R \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1\lambda}^R & a_{m2\lambda}^R & \cdots & a_{mn\lambda}^R \end{bmatrix}$$

$$b_\lambda^L = \left[b_{1\lambda}^L, b_{2\lambda}^L, \cdots, b_{m\lambda}^L\right]^T, \qquad b_\lambda^R = \left[b_{1\lambda}^R, b_{2\lambda}^R, \cdots, b_{m\lambda}^R\right]^T$$

In the following, we introduce the concepts of optimal solutions of the MOLP problem.

DEFINITION 4.

A point $x^* \in R^n$ is said to be a complete optimal solution to the MOLP problem if it holds that $(\langle c_\lambda^L, x^* \rangle, \langle c_\lambda^R, x^* \rangle)^T \geqq (\langle c_\lambda^L, x \rangle, \langle c_\lambda^R, x \rangle)^T$, for all $x \in X = \{x; A_\lambda^L x \leqq b_\lambda^L, A_\lambda^R x \leqq b_\lambda^R, x \geqq 0, \lambda \in [0,1]\}$ and $\lambda \in [0, 1]$.

DEFINITION 5.

A point $x^* \in R^n$ is said to be a Pareto optimal solution to the MOLP problem if there is no $x \in X$ such that $(\langle c_\lambda^L, x^* \rangle, \langle c_\lambda^R, x^* \rangle)^T \leq (\langle c_\lambda^L, x \rangle, \langle c_\lambda^R, x \rangle)^T$, $\lambda \in [0, 1]$ holds.

DEFINITION 6.

A point $x^* \in R^n$ is said to be a weak Pareto optimal solution to the MOLP problem if there is no $x \in X$ such that $(\langle c_\lambda^L, x^* \rangle, \langle c_\lambda^R, x^* \rangle)^T < (\langle c_\lambda^L, x \rangle, \langle c_\lambda^R, x \rangle)^T$, $\lambda \in [0, 1]$ holds.

THEOREM 7.

Let $x^* \in R^n$ be a feasible solution to the FMOLP problem. Then

1. x^* is a complete optimal solution to the FMOLP problem, if and only if x^* is a complete optimal solution to the MOLP problem.

2. x^* is a Pareto optimal solution to the FMOLP problem, if and only if x^* is a Pareto optimal solution to the MOLP problem.
3. x^* is a weak Pareto optimal solution to the FMOLP problem, if and only if x^* is a weak Pareto optimal solution to the MOLP problem.

Proof.

The proof follows directly from Definitions 1−6.

2.2 Model and Pareto Optimal Solution for FMOLP$_\alpha$ Problems

Obviously, a feasible solution must satisfy the constraints for all $\lambda \in [0, 1]$. However, this is a too strong condition to get an optimal solution. We therefore consider a typical parameter c_i represented by a fuzzy number $\tilde{c}_i$. The possibility of such a parameter c_i taking values in the range $[c_{i\lambda}^L, c_{i\lambda}^R]$ is λ or above. While the possibility of c_i taking values beyond $[c_{i\lambda}^L, c_{i\lambda}^R]$ is less than λ. Thus, one would be generally more interested in a solution using parameters c_i taking values in $[c_{i\lambda}^L, c_{i\lambda}^R]$ with $\lambda \geq \alpha > 0$. As a special case, if the parameters involved are either a real number or a fuzzy number with a triangular membership function, then, we will have the usual non-fuzzy optimization problem (suppose we choose $\alpha = 1$). To formulate this idea, we introduce the following FMOLP$_\alpha$ model.

$$(\text{FMOLP}_\alpha)\quad \begin{cases} \text{Maximize} & \langle \tilde{c}, x \rangle_F = \sum_{i=1}^{n} \tilde{c}_i x_i \\ \text{subject to} & \tilde{A}x \preceq_{\alpha} \tilde{b},\ x \geq 0, \end{cases} \quad 0 \leq \alpha \leq 1 \tag{3}$$

where

$$\tilde{c} = \begin{pmatrix} \tilde{c}_{11} & \tilde{c}_{12} & \cdots & \tilde{c}_{1n} \\ \tilde{c}_{21} & \tilde{c}_{22} & \cdots & \tilde{c}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{c}_{k1} & \tilde{c}_{k2} & \cdots & \tilde{c}_{kn} \end{pmatrix}, \tilde{A} = \begin{pmatrix} \tilde{a}_{11} & \tilde{a}_{12} & \cdots & \tilde{a}_{1n} \\ \tilde{a}_{21} & \tilde{a}_{22} & \cdots & \tilde{a}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{a}_{m1} & \tilde{a}_{m2} & \cdots & \tilde{a}_{mn} \end{pmatrix},$$

$$\tilde{b} = (\tilde{b}_1, \tilde{b}_2, \cdots, \tilde{b}_m)^T \in F^*(R^m),$$

and $\tilde{c}_{sj}, \tilde{a}_{ij} \in F^*(R), s = 1, 2, \cdots, k, i = 1, 2, \cdots, m, j = 1, 2, \cdots, n.$

Now, associated with the $FMOLP_\alpha$ problem, consider the following $MOLP_\alpha$ problem,

$$(MOLP_\alpha)\quad \begin{cases} \text{maximize} & \left(\langle c_\lambda^L, x\rangle, \langle c_\lambda^R, x\rangle\right)^T \\ \text{subject to} & A_\lambda^L x \leqq b_\lambda^L,\ A_\lambda^R x \leqq b_\lambda^R,\ x \geqq 0,\ \forall \lambda \in [\alpha, 1] \end{cases} \tag{4}$$

where

$$C_\lambda^L = \begin{bmatrix} c_{11\lambda}^L & c_{12\lambda}^L & \cdots & c_{1n\lambda}^L \\ c_{21\lambda}^L & c_{22\lambda}^L & \cdots & c_{2n\lambda}^L \\ \vdots & \vdots & \ddots & \vdots \\ c_{k1\lambda}^L & c_{k2\lambda}^L & \cdots & c_{kn\lambda}^L \end{bmatrix},\quad C_\lambda^R = \begin{bmatrix} c_{11\lambda}^R & c_{12\lambda}^R & \cdots & c_{1n\lambda}^R \\ c_{21\lambda}^R & c_{22\lambda}^R & \cdots & c_{2n\lambda}^R \\ \vdots & \vdots & \ddots & \vdots \\ c_{k1\lambda}^R & c_{k2\lambda}^R & \cdots & c_{kn\lambda}^R \end{bmatrix}$$

$$A_\lambda^L = \begin{bmatrix} a_{11\lambda}^L & a_{12\lambda}^L & \cdots & a_{1n\lambda}^L \\ a_{21\lambda}^L & a_{22\lambda}^L & \cdots & a_{2n\lambda}^L \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1\lambda}^L & a_{m2\lambda}^L & \cdots & a_{mn\lambda}^L \end{bmatrix},\ A_\lambda^R = \begin{bmatrix} a_{11\lambda}^R & a_{12\lambda}^R & \cdots & a_{1n\lambda}^R \\ a_{21\lambda}^R & a_{22\lambda}^R & \cdots & a_{2n\lambda}^R \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1\lambda}^R & a_{m2\lambda}^R & \cdots & a_{mn\lambda}^R \end{bmatrix}$$

$$b_\lambda^L = \left[b_{1\lambda}^L, b_{2\lambda}^L, \cdots, b_{m\lambda}^L\right]^T,\qquad b_\lambda^R = \left[b_{1\lambda}^R, b_{2\lambda}^R, \cdots, b_{m\lambda}^R\right]^T.$$

In the following, we introduce the concepts of optimal solutions of the $MOLP_\alpha$ problem.

DEFINITION 8.

A point $x^* \in R^n$ is said to be a complete optimal solution to the FMOLP problem if it holds that $\langle \tilde{c}, x^*\rangle_F \succeq_\alpha \langle \tilde{c}, x\rangle_F$ for all $x \in \tilde{X}_\alpha$.

DEFINITION 9.

A point $x^* \in R^n$ is said to be a Pareto optimal solution to the FMOLP problem if there is no $x \in \tilde{X}_\alpha$ such that $\langle \tilde{c}, x\rangle_F \succeq_\alpha \langle \tilde{c}, x^*\rangle_F$ holds.

DEFINITION 10.

A point $x^* \in R^n$ is said to be a weak Pareto optimal solution to the FMOLP problem if there is no $x \in \tilde{X}_\alpha$ such that $\langle \tilde{c}, x\rangle_F \succ_\alpha \langle \tilde{c}, x^*\rangle_F$ holds.

DEFINITION 11.

A point $x^* \in R^n$ is said to be a complete optimal solution to the MOLP$_\alpha$ problem if it holds that $(\langle c_\lambda^L, x^* \rangle, \langle c_\lambda^R, x^* \rangle)^T \geqq (\langle c_\lambda^L, x \rangle, \langle c_\lambda^R, x \rangle)^T$, for all $x \in X_\alpha = \{x;\ A_\lambda^L x \leqq b_\lambda^L,\ A_\lambda^R x \leqq b_\lambda^R,\ x \geqq 0,\ \lambda \in [\alpha, 1]\}$ and $\lambda \in [\alpha, 1]$.

DEFINITION 12.

A point $x^* \in R^n$ is said to be a Pareto optimal solution to the MOLP$_\alpha$ problem if there is no $x \in X_\alpha$ such that $(\langle c_\lambda^L, x^* \rangle, \langle c_\lambda^R, x^* \rangle)^T \leq (\langle c_\lambda^L, x \rangle, \langle c_\lambda^R, x \rangle)^T$, $\lambda \in [\alpha, 1]$ holds.

DEFINITION 13.

A point $x^* \in R^n$ is said to be a weak Pareto optimal solution to the MOLP$_\alpha$ problem if there is no $x \in X_\alpha$ such that $(\langle c_\lambda^L, x^* \rangle, \langle c_\lambda^R, x^* \rangle)^T < (\langle c_\lambda^L, x \rangle, \langle c_\lambda^R, x \rangle)^T$, $\lambda \in [\alpha, 1]$ holds.

THEOREM 14.

Let $x^* \in R^n$ be a feasible solution to the FMOLP$_\alpha$ problem. Then

1. x^* is a complete optimal solution to the FMOLP$_\alpha$ problem, if and only if x^* is a complete optimal solution to the MOLP$_\alpha$ problem.
2. x^* is a Pareto optimal solution to the FMOLP$_\alpha$ problem, if and only if x^* is a Pareto optimal solution to the MOLP$_\alpha$ problem.
3. x^* is a weak Pareto optimal solution to the FMOLP$_\alpha$ problem, if and only if x^* is a weak Pareto optimal solution to the MOLP$_\alpha$ problem.

Proof.

The proof follows directly from Definitions 8−13 and Theorem 14.

In this section, we have addressed the FMOLP problem and have introduced the concepts of complete optimal solution, Pareto optimal solution, and weak Pareto optimal solution for FMOLP, FMOLP$_\alpha$, MOLP, and MOLP$_\alpha$. We have also proposed an efficient approach for solving the FMOLP and FMOLP$_\alpha$ problems, which is to transform them into the associative crisp MOLP and MOLP$_\alpha$.

3. SOLUTION TRANSFORMATION THEORIES FOR FUZZY MULTI-OBJECTIVE DECISION-MAKING PROBLEMS

As outlined in Section 2, the possible values of parameters in the FMOLP are appropriate to be represented by fuzzy numbers. Here we will show how a fuzzy number parameters-based FMOLP problem is transformed into an associated MOLP problem.

3.1 General MOLP Transformation Theories

Consider the situation in which all parameters of the objective functions and the constraints are fuzzy numbers represented in any form of membership functions. Such FMOLP problems can be formulated as follows:

Lemma 15. If a fuzzy set $\tilde{c}$ on R has a trapezoidal membership function (see Figure 1):

$$\mu_{\tilde{c}}(x) = \begin{cases} 0 & x < c_\beta^L \\ \dfrac{\alpha-\beta}{c_\alpha^L - c_\beta^L}\left(x - c_\beta^L\right) + \beta & c_\beta^L \leqq x < c_\alpha^L \\ \alpha & c_\alpha^L \leqq x \leqq c_\alpha^R \\ \dfrac{\alpha-\beta}{c_\alpha^R - c_\beta^R}\left(x - c_\beta^R\right) + \beta & c_\alpha^R < x \leqq c_\beta^R \\ 0 & c_\beta^R < x \end{cases}$$

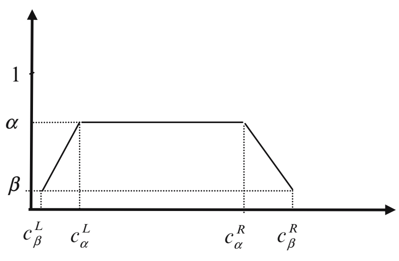

Figure 1. Trapezoidal membership function

and there is $x^* \in X^n$ such that $\left\langle c_\beta^L, x \right\rangle \leqq \left\langle c_\beta^L, x^* \right\rangle$, $\left\langle c_\alpha^L, x \right\rangle \leqq \left\langle c_\alpha^L, x^* \right\rangle$, $\left\langle c_\beta^R, x \right\rangle \leqq \left\langle c_\beta^R, x^* \right\rangle$, $(0 \le \beta < \alpha \le 1)$, and $\left\langle c_\alpha^R, x \right\rangle \leqq \left\langle c_\alpha^R, x^* \right\rangle$, for any $x \in X^n$, then

$$\left\langle c_\lambda^L, x \right\rangle \leqq \left\langle c_\lambda^L, x^* \right\rangle$$

$$\left\langle c_\lambda^R, x \right\rangle \leqq \left\langle c_\lambda^R, x^* \right\rangle$$

for any $\lambda \in [\beta, \alpha]$.

Proof.
As a λ-section of the trapezoidal fuzzy set $\tilde{c}$ is

$$c_\lambda^L = \frac{\lambda-\beta}{\alpha-\beta}\left(c_\alpha^L - c_\beta^L\right) + c_\beta^L \text{ and } c_\lambda^R = \frac{\lambda-\beta}{\alpha-\beta}\left(c_\alpha^R - c_\beta^R\right) + c_\beta^R$$

Therefore, we have

$$\begin{aligned}
\left\langle c_\lambda^L, x \right\rangle &= \left\langle \frac{\lambda-\beta}{\alpha-\beta}\left(c_\alpha^L - c_\beta^L\right), x \right\rangle + \left\langle c_\beta^L, x \right\rangle \\
&= \frac{\lambda-\beta}{\alpha-\beta}\left(\left\langle c_\alpha^L, x \right\rangle - \left\langle c_\beta^L, x \right\rangle\right) + \left\langle c_\beta^L, x \right\rangle \\
&= \frac{\lambda-\beta}{\alpha-\beta}\left\langle c_\alpha^L, x \right\rangle + \frac{\alpha-\lambda}{\alpha-\beta}\left\langle c_\beta^L, x \right\rangle \\
&\leqq \frac{\lambda-\beta}{\alpha-\beta}\left\langle c_\alpha^L, x^* \right\rangle + \frac{\alpha-\lambda}{\alpha-\beta}\left\langle c_1^L, x^* \right\rangle \\
&= \left\langle \frac{\lambda-\beta}{\alpha-\beta}\left(c_\alpha^L - c_\beta^L\right), x^* \right\rangle + \left\langle c_\beta^L, x^* \right\rangle = \left\langle c_\lambda^L, x^* \right\rangle
\end{aligned}$$

from $\left\langle c_1^L, x \right\rangle \leqq \left\langle c_1^L, x^* \right\rangle$, $\left\langle c_\alpha^L, x \right\rangle \leqq \left\langle c_\alpha^L, x^* \right\rangle$ and $0 \le \beta \le \lambda \le \alpha \le 1$, we can prove $\left\langle c_\lambda^R, x \right\rangle \leqq \left\langle c_\lambda^R, x^* \right\rangle$ from a similar reason.

THEOREM 16.

If all the fuzzy parameters $\tilde{c}_{sj}, \tilde{a}_{ij}$, and $\tilde{b}_i$ have trapezoidal membership functions:

$$\mu_{\tilde{z}}(t) = \begin{cases} 0 & t < z_\beta^L \\ \dfrac{\alpha-\beta}{z_\alpha^L - z_\beta^L}\left(t - z_\beta^L\right) + \beta & z_\beta^L \leqq t < z_\alpha^L \\ \alpha & z_\alpha^L \leqq t < z_\alpha^R \\ \dfrac{\alpha-\beta}{z_\beta^R - z_\alpha^R}\left(-t + z_\beta^R\right) + \beta & z_\alpha^R \leqq t \leqq z_\beta^R \\ 0 & z_\beta^R < t \end{cases} \tag{5}$$

where $\tilde{z}$ denotes $\tilde{c}_{sj}, \tilde{a}_{ij}$ or $\tilde{b}_i$ respectively, then the space of feasible solutions X is defined by the set of $x \in X$ with x_i, for $i = 1,2,\ldots,n$ satisfying

$$\begin{cases} \sum_{j=1}^{n} a_{ij}{}_\alpha^L x_j \leqq b_i{}_\alpha^L \\ \sum_{j=1}^{n} a_{ij}{}_\alpha^R x_j \leqq b_i{}_\alpha^R \\ \sum_{j=1}^{n} a_{ij}{}_\beta^L x_j \leqq b_i{}_\beta^L \ . \\ \sum_{j=1}^{n} a_{ij}{}_\beta^R x_j \leqq b_i{}_\beta^R \\ x_i \geqq 0 \end{cases} \tag{6}$$

Proof.

From Theorem 7, X is defined by

$$X = \{x \in R^n \mid \sum_{j=1}^{n} a_{ij\lambda}^{L} x_j \leqq b_{i\lambda}^{L}, \sum_{j=1}^{n} a_{ij\lambda}^{R} x_j \leqq b_{i\lambda}^{R}, x \geqq 0 \\ \forall \lambda \in [\beta, \alpha] \text{ and } \mathrm{i} = 1, 2, \cdots, m\}. \tag{7}$$

That is, X is the set of $x \in R^n$ with $x \geqq 0$ and satisfying

$$I_{i\lambda} = \sum_{j=1}^{n} a_{ij\lambda}^{L} x_j - b_{i\lambda}^{L} \leqq 0,\ J_{i\lambda} = \sum_{j=1}^{n} a_{ij\lambda}^{R} x_j - b_{i\lambda}^{R} \leqq 0 \\ \forall \lambda \in [\beta, \alpha] \text{ and } i = 1, 2, \cdots, m. \tag{8}$$

For the fuzzy sets with trapezoidal membership functions, we have

$$a_{ij\lambda}^{L} = \frac{a_{ij\alpha}^{L} - a_{ij\beta}^{L}}{\alpha - \beta}(\lambda - \beta) + a_{ij\beta}^{L} \\ a_{ij\lambda}^{R} = \frac{a_{ij\alpha}^{R} - a_{ij\beta}^{R}}{\alpha - \beta}(\lambda - \beta) + a_{ij\beta}^{R}, \tag{9}$$

$$b_{i\lambda}^{L} = \frac{b_{i\alpha}^{L} - b_{i\beta}^{L}}{\alpha - \beta}(\lambda - \beta) + b_{i\beta}^{L} \\ b_{i\lambda}^{R} = \frac{b_{i\alpha}^{R} - b_{i\beta}^{R}}{\alpha - \beta}(\lambda - \beta) + b_{i\beta}^{R} \tag{10}$$

Substituting Eqs. (9) and (10) into (8), we have

$$I_{i\lambda} = \sum_{j=1}^{n} [\frac{a_{ij\alpha}^{L} - a_{ij\beta}^{L}}{\alpha - \beta}(\lambda - \beta) + a_{ij\beta}^{L}] x_j - [\frac{b_{i\alpha}^{L} - b_{i\beta}^{L}}{\alpha - \beta}(\lambda - \beta) + b_{i\beta}^{L}] \\ = \frac{\lambda - \beta}{\alpha - \beta}\left(\sum_{j=1}^{n} a_{ij\alpha}^{L} x_j - b_{i\alpha}^{L}\right) + \frac{\alpha - \lambda}{\alpha - \beta}\left(\sum_{j=1}^{n} a_{ij\beta}^{L} x_j - b_{i\beta}^{L}\right) \tag{11}$$

$$J_{i\lambda} = \sum_{j=1}^{n}[\frac{a_{ij}{}_{\alpha}^{R} - a_{ij}{}_{\beta}^{R}}{\alpha-\beta}(\lambda-\beta) + a_{ij}{}_{\beta}^{R}]x_j - [\frac{b_i{}_{\alpha}^{R} - b_i{}_{\beta}^{R}}{\alpha-\beta}(\lambda-\beta) + b_i{}_{\beta}^{R}]$$
$$= \frac{\lambda-\beta}{\alpha-\beta}\left(\sum_{j=1}^{n} a_{ij}{}_{\alpha}^{R} x_j - b_i{}_{\alpha}^{R}\right) + \frac{\alpha-\lambda}{\alpha-\beta}\left(\sum_{j=1}^{n} a_{ij}{}_{\beta}^{R} x_j - b_i{}_{\beta}^{R}\right) \tag{12}$$

Now, our problem becomes to show that $I_{i\lambda} \leqq 0$, $J_{i\lambda} \leqq 0$, $\forall\lambda \in [\beta,\alpha]$ and $i = 1,2,\cdots,m$ if (6) is satisfied. From Eq. (6), we have

$$\begin{cases} \sum_{j=1}^{n} a_{ij\alpha}^{L} x_j - b_{i\alpha}^{L} \leqq 0 & \text{(13a)} \\ \sum_{j=1}^{n} a_{ij\alpha}^{R} x_j - b_{i\alpha}^{R} \leqq 0 & \text{(13b)} \\ \sum_{j=1}^{n} a_{ij\beta}^{L} x_j - b_{i\beta}^{L} \leqq 0 & \text{(13c)} \\ \sum_{j=1}^{n} a_{ij\beta}^{R} x_j - b_{i\beta}^{R} \leqq 0. & \text{(13d)} \end{cases}$$

Thus, from Eqs. (13a) and (13c), we have for any $\lambda \in [\beta, \alpha]$ and $i = 1,2,\ldots,m$

$$I_{i\lambda} = \frac{\lambda-\beta}{\alpha-\beta}\left(\sum_{j=1}^{n} a_{ij}{}_{\alpha}^{L} x_j - b_i{}_{\alpha}^{L}\right) + \frac{\alpha-\lambda}{\alpha-\beta}\left(\sum_{j=1}^{n} a_{ij}{}_{\beta}^{L} x_j - b_i{}_{\beta}^{L}\right) \leqq 0$$

and from Eqs. (13b) and (13d), we have for any $\lambda \in [\beta, \alpha]$ and $i = 1,2,\ldots,m$

$$J_{i\lambda} = \frac{\lambda-\beta}{\alpha-\beta}\left(\sum_{j=1}^{n} a_{ij}{}_{\alpha}^{R} x_j - b_i{}_{\alpha}^{R}\right) + \frac{\alpha-\lambda}{\alpha-\beta}\left(\sum_{j=1}^{n} a_{ij}{}_{\beta}^{R} x_j - b_i{}_{\beta}^{R}\right) \leqq 0.$$

Corollary 17.

If all the fuzzy parameters $\tilde{c}_{sj}, \tilde{a}_{ij}$ and $\tilde{b}_i$ have piece-wise trapezoidal membership functions

$$\mu_{\tilde{z}}(t)=\begin{cases} 0 & t < z_{\alpha 0}^{L} \\ \dfrac{\alpha_1-\alpha_0}{z_{\alpha 1}^{L}-z_{\alpha 0}^{L}}\left(t-z_{\alpha 0}^{L}\right)+\alpha_0 & z_{\alpha 0}^{L} \leqq t < z_{\alpha 1}^{L} \\ \dfrac{\alpha_2-\alpha_1}{z_{\alpha 2}^{L}-z_{\alpha 1}^{L}}\left(t-z_{\alpha 1}^{L}\right)+\alpha_1 & z_{\alpha 1}^{L} \leqq t < z_{\alpha 2}^{L} \\ \cdots & \cdots \\ \alpha & z_{\alpha n}^{L} \leqq t < z_{\alpha n}^{R} \\ \dfrac{\alpha_n-\alpha_{n-1}}{z_{\alpha n-1}^{R}-z_{\alpha n}^{R}}\left(-t+z_{\alpha n-1}^{R}\right)+\alpha_{n-1} & z_{\alpha n}^{R} \leqq t < z_{\alpha n-1}^{R} \\ \cdots & \cdots \\ \dfrac{\alpha_0-\alpha_1}{z_{\alpha 1}^{R}-z_{\alpha 0}^{R}}\left(-t+z_{\alpha 0}^{R}\right)+\alpha_0 & z_{\alpha 1}^{R} \leqq t \leqq z_{\alpha 0}^{R} \\ 0 & z_{\alpha 0}^{R} < t \end{cases} \tag{14}$$

where $\tilde{z}$ denotes $\tilde{c}_{sj}, \tilde{a}_{ij}$ or $\tilde{b}_i$ respectively, then the space of feasible solutions X is defined by the set of $x \in X$ with x_i, for $i = 1,2,\ldots,n$ satisfying

$$\begin{cases} \sum_{j=1}^{n} a_{ij}{}_{\alpha 0}^{L} x_j \leqq b_i{}_{\alpha 0}^{L} \\ \sum_{j=1}^{n} a_{ij}{}_{\alpha 0}^{R} x_j \leqq b_i{}_{\alpha 0}^{R} \\ \sum_{j=1}^{n} a_{ij}{}_{\alpha 1}^{L} x_j \leqq b_i{}_{\alpha 1}^{L} \\ \sum_{j=1}^{n} a_{ij}{}_{\alpha 1}^{R} x_j \leqq b_i{}_{\alpha 1}^{R} \\ \vdots \\ \sum_{j=1}^{n} a_{ij}{}_{\alpha n}^{L} x_j \leqq b_i{}_{\alpha n}^{L} \\ \sum_{j=1}^{n} a_{ij}{}_{\alpha n}^{R} x_j \leqq b_i{}_{\alpha n}^{R} \\ x_i \geqq 0 \end{cases} . \tag{15}$$

THEOREM 18.

Let all the fuzzy parameters be piece-wise trapezoidal membership functions in $FMOLP_\alpha$:

$$\mu_{\tilde{z}}(t) = \begin{cases} 0 & t < z^L_{\alpha 0} \\ \dfrac{\alpha_1 - \alpha_0}{z^L_{\alpha 1} - z^L_{\alpha 0}}\left(t - z^L_{\alpha 0}\right) + \alpha_0 & z^L_{\alpha 0} \leqq t < z^L_{\alpha 1} \\ \dfrac{\alpha_2 - \alpha_1}{z^L_{\alpha 2} - z^L_{\alpha 1}}\left(t - z^L_{\alpha 1}\right) + \alpha_1 & z^L_{\alpha 1} \leqq t < z^L_{\alpha 2} \\ \dots & \dots \\ 1 & z^L_{\alpha n} \leqq t < z^R_{\alpha n} \\ \dfrac{\alpha_n - \alpha_{n-1}}{z^R_{\alpha n-1} - z^R_{\alpha n}}\left(-t + z^R_{\alpha n-1}\right) + \alpha_{n-1} & z^R_{\alpha n} \leqq t < z^R_{\alpha n-1} \\ \dots & \dots \\ \dfrac{\alpha_0 - \alpha_1}{z^R_{\alpha 1} - z^R_{\alpha 0}}\left(-t + z^R_{\alpha 0}\right) + \alpha_0 & z^R_{\alpha 1} \leqq t \leqq z^R_{\alpha 0} \\ 0 & z^R_{\alpha 0} < t \end{cases} \tag{16}$$

If a point $x^* \in R^n$ be a feasible solution to the $FMOLP_\alpha$ problem, then x^* is a complete optimal solution to the problem if and only if x^* is a complete optimal solution to the $MOLP_\alpha$ problem:

$$
(\text{MOLP}_{\alpha}) \quad \begin{cases}
\text{Maximize} & \left\langle c^{L}_{\alpha 0}, x \right\rangle \\
& \left\langle c^{R}_{\alpha 0}, x \right\rangle \\
& \left\langle c^{L}_{\alpha 1}, x \right\rangle \\
& \vdots \\
& \left\langle c^{R}_{\alpha n-1}, x \right\rangle \\
& \left\langle c^{L}_{\alpha n}, x \right\rangle \\
& \left\langle c^{R}_{\alpha n}, x \right\rangle \\
\text{subject to} & \sum_{j=1}^{n} a_{ij}{}^{L}_{\alpha 0}\, x_j \leqq b_i{}^{L}_{\alpha 0} \\
& \sum_{j=1}^{n} a_{ij}{}^{R}_{\alpha 0}\, x_j \leqq b_i{}^{R}_{\alpha 0} \\
& \sum_{j=1}^{n} a_{ij}{}^{L}_{\alpha 1}\, x_j \leqq b_i{}^{L}_{\alpha 1} \\
& \vdots \\
& \sum_{j=1}^{n} a_{ij}{}^{R}_{\alpha 1}\, x_j \leqq b_i{}^{R}_{\alpha 1} \\
& \sum_{j=1}^{n} a_{ij}{}^{L}_{\alpha n}\, x_j \leqq b_i{}^{L}_{\alpha n} \\
& \sum_{j=1}^{n} a_{ij}{}^{R}_{\alpha n}\, x_j \leqq b_i{}^{R}_{\alpha n} \\
& x_i \geqq 0
\end{cases} \tag{17}
$$

where $\alpha = \alpha_0 \leq \alpha_1 \leq \cdots \leq \alpha_{n-1} \leq \alpha_n = 1.$

Proof.

If x^* is an optimal solution to the FMOLP$_\alpha$ problem, then for any $x \in \widetilde{X}_\alpha$, we have $\left\langle \widetilde{c}, x^* \right\rangle_F \succeq_\alpha \left\langle \widetilde{c}, x \right\rangle_F$. Therefore, for any $\lambda \in [\alpha, 1]$,

$$
\left(\sum_{i=1}^{n} \widetilde{c}_i x_i^*\right)^L_\lambda \geqq \left(\sum_{i=1}^{n} \widetilde{c}_i x_i\right)^L_\lambda \quad \text{and} \quad \left(\sum_{i=1}^{n} \widetilde{c}_i x_i^*\right)^R_\lambda \geqq \left(\sum_{i=1}^{n} \widetilde{c}_i x_i\right)^R_\lambda
$$

that is

$$\sum_{i=1}^{n} c_{i\lambda}^{L} x_i^{*} \geqq \sum_{i=1}^{n} c_{i\lambda}^{L} x_i \quad \text{and} \quad \sum_{i=1}^{n} c_{i\lambda}^{R} x_i^{*} \geqq \sum_{i=1}^{n} c_{i\lambda}^{R} x_i$$

Hence x^* is a complete optimal solution to the MOLP$_\alpha$ problem by Definition 11. □

If x^* is a complete optimal solution to the MOLP$_\alpha$ problem, then for all $x \in X_\alpha$, we have

$$(\langle c_{\alpha i}^{L}, x^{*}\rangle, \langle c_{\alpha i}^{R}, x^{*}\rangle)^{T} \geqq (\langle c_{\alpha i}^{L}, x\rangle, \langle c_{\alpha i}^{R}, x\rangle)^{T}, \quad i = 0, 1, \cdots, n$$

that is

$$\sum_{j=1}^{n} c_{j\alpha i}^{L} x_j^{*} \geqq \sum_{j=1}^{n} c_{j\alpha i}^{L} x_j, \; \sum_{j=1}^{n} c_{j\alpha i}^{R} x_j^{*} \geqq \sum_{j=1}^{n} c_{j\alpha i}^{R} x_j, \; i = 1, 2, \cdots, n$$

For any $\lambda \in [\alpha, 1]$, there exist $i \in \{1, 2, \cdots, n\}$ so that $\lambda \in [\alpha_{i-1}, \alpha_i]$.

As $\tilde{c}$ has a piece-wise trapezoidal membership function, we have

$$c_{\lambda}^{L} = \frac{\lambda - \alpha_{i-1}}{\alpha_i - \alpha_{i-1}} \left(c_{\alpha 1}^{L} - c_{\alpha i-1}^{L} \right) + c_{\alpha i-1}^{L} \text{ and}$$

$$c_{\lambda}^{R} = \frac{\lambda - \alpha_{i-1}}{\alpha_i - \alpha_{i-1}} \left(c_{\alpha i}^{R} - c_{\alpha i-1}^{R} \right) + c_{\alpha i-1}^{R}$$

From Lemma 15, we have

$$\sum_{i=1}^{n} c_{i\lambda}^{L} x_i^{*} \geqq \sum_{i=1}^{n} c_{i\lambda}^{L} x_i \quad \text{and} \quad \sum_{i=1}^{n} c_{i\lambda}^{R} x_i^{*} \geqq \sum_{i=1}^{n} c_{i\lambda}^{R} x_i,$$

for any $\lambda \in [\alpha, 1]$. Therefore, x^* is an optimal solution to the FMOLP$_\alpha$ problem.

THEOREM 19.

Let all the fuzzy parameters be piece-wise trapezoidal membership functions in FMOLP$_\alpha$:

$$\mu_{\tilde{z}}(t) = \begin{cases} 0 & t < z_{\alpha 0}^L \\ \dfrac{\alpha_1 - \alpha_0}{z_{\alpha 1}^L - z_{\alpha 0}^L}\left(t - z_{\alpha 0}^L\right) + \alpha_0 & z_{\alpha 0}^L \leqq t < z_{\alpha 1}^L \\ \dfrac{\alpha_2 - \alpha_1}{z_{\alpha 2}^L - z_{\alpha 1}^L}\left(t - z_{\alpha 1}^L\right) + \alpha_1 & z_{\alpha 1}^L \leqq t < z_{\alpha 2}^L \\ \cdots & \cdots \\ 1 & z_{\alpha n}^L \leqq t < z_{\alpha n}^R \\ \dfrac{\alpha_n - \alpha_{n-1}}{z_{\alpha n-1}^R - z_{\alpha n}^R}\left(-t + z_{\alpha n-1}^R\right) + \alpha_{n-1} & z_{\alpha n}^R \leqq t < z_{\alpha n-1}^R \\ \cdots & \cdots \\ \dfrac{\alpha_0 - \alpha_1}{z_{\alpha 1}^R - z_{\alpha 0}^R}\left(-t + z_{\alpha 0}^R\right) + \alpha_0 & z_{\alpha 1}^R \leqq t \leqq z_{\alpha 0}^R \\ 0 & z_{\alpha 0}^R < t \end{cases} . \quad (18)$$

Let a point $x^* \in \tilde{X}_\alpha$ be any feasible solution to the FMOLP$_\alpha$ problem. Then x^* is a Pareto optimal solution to the problem if and only if x^* is a Pareto optimal solution to the MOLP$_\alpha$ problem:

$$
(\text{MOLP}_{\alpha})\quad
\begin{cases}
\text{Maximize} & \left\langle c^{L}_{\alpha 0}, x\right\rangle \\
& \left\langle c^{R}_{\alpha 0}, x\right\rangle \\
& \left\langle c^{L}_{\alpha 1}, x\right\rangle \\
& \vdots \\
& \left\langle c^{R}_{\alpha n-1}, x\right\rangle \\
& \left\langle c^{L}_{\alpha n}, x\right\rangle \\
& \left\langle c^{R}_{\alpha n}, x\right\rangle \\
\text{subject to} & \sum_{j=1}^{n} a_{ij}{}^{L}_{\alpha 0} x_j \leqq b_i{}^{L}_{\alpha 0} \\
& \sum_{j=1}^{n} a_{ij}{}^{R}_{\alpha 0} x_j \leqq b_i{}^{R}_{\alpha 0} \\
& \sum_{j=1}^{n} a_{ij}{}^{L}_{\alpha 1} x_j \leqq b_i{}^{L}_{\alpha 1} \\
& \vdots \\
& \sum_{j=1}^{n} a_{ij}{}^{R}_{\alpha 1} x_j \leqq b_i{}^{R}_{\alpha 1} \\
& \sum_{j=1}^{n} a_{ij}{}^{L}_{\alpha n} x_j \leqq b_i{}^{L}_{\alpha n} \\
& \sum_{j=1}^{n} a_{ij}{}^{R}_{\alpha n} x_j \leqq b_i{}^{R}_{\alpha n} \\
& x_i \geqq 0
\end{cases}
\tag{19}
$$

where $\alpha = \alpha_0 \le \alpha_1 \le \cdots \le \alpha_{n-1} \le \alpha_n = 1$.

Proof.

Let $x^* \in \widetilde{X}_\alpha$ be a Pareto optimal solution to the FMOLP$_\alpha$ problem. On the contrary, we suppose that there exists an $\overline{x} \in X$ such that

$$(\left\langle c^{L}_{\alpha_i},\ x^*\right\rangle,\ \left\langle c^{R}_{\alpha_i},\ x^*\right\rangle)^{T} \le (\left\langle c^{L}_{\alpha_i},\ \overline{x}\right\rangle,\ \left\langle c^{R}_{\alpha_i},\ \overline{x}\right\rangle)^{T},\ i = 0,\ 1,\cdots,\ n \tag{20}$$

Therefore

$$0 \le (\left\langle c^{L}_{\alpha_i}, \overline{x}\right\rangle - \left\langle c^{L}_{\alpha_i}, x^*\right\rangle,\ \left\langle c^{R}_{\alpha_i}, \overline{x}\right\rangle - \left\langle c^{R}_{\alpha_i}, x^*\right\rangle,)^{T},\ i = 0,\ 1,\ 2,\ \cdots,\ n \tag{21}$$

Hence

$$0 \le \left\langle c^{L}_{\alpha_i}, \bar{x} \right\rangle - \left\langle c^{L}_{\alpha_i}, x^* \right\rangle,\ 0 \le \left\langle c^{R}_{\alpha_i}, \bar{x} \right\rangle - \left\langle c^{R}_{\alpha_i}, x^* \right\rangle,\ i = 0, 1, 2, \cdots, n \quad (22)$$

That is

$$\left\langle c^{L}_{\alpha i}, \bar{x} \right\rangle \ge \left\langle c^{L}_{\alpha i}, x^* \right\rangle, \left\langle c^{R}_{\alpha i}, \bar{x} \right\rangle \ge \left\langle c^{R}_{\alpha 1}, x^* \right\rangle, \quad i = 1, 2, \cdots, n$$

By using Lemma 15, for any $\lambda \in [\alpha, 1]$, we have

$$\left\langle c^{L}_{\lambda}, x^* \right\rangle \le \left\langle c^{L}_{\lambda}, \bar{x} \right\rangle \quad \text{and} \quad \left\langle c^{R}_{\lambda}, x^* \right\rangle \le \left\langle c^{R}_{\lambda}, \bar{x} \right\rangle$$

that is $\left\langle \tilde{c}, \bar{x} \right\rangle_F \succeq \left\langle \tilde{c}, x^* \right\rangle_F$. However, this contradicts the assumption that $x^* \in \tilde{X}_\alpha$ is a Pareto optimal solution to the FMOLP problem.

Let $x \in X_\alpha$ be a Pareto optimal solution to the MOLP$_\alpha$ problem. If x^* is not a Pareto optimal solution to the problem, then there exists an $\bar{x} \in \tilde{X}_\alpha$ such that $\left\langle \tilde{c}, \bar{x} \right\rangle_F \succeq \left\langle \tilde{c}, x^* \right\rangle_F$. Therefore, for any $\lambda \in [\alpha, 1]$, we have

$$\left(\sum_{i=1}^{n} \tilde{c}_i x_i^*\right)^L_\lambda \le \left(\sum_{i=1}^{n} \tilde{c}_i \bar{x}_i\right)^L_\lambda \quad \text{and} \quad \left(\sum_{i=1}^{n} \tilde{c}_i x_i^*\right)^R_\lambda \le \left(\sum_{i=1}^{n} \tilde{c}_i \bar{x}_i\right)^R_\lambda$$

That is

$$\left\langle c^{L}_{\lambda}, x^* \right\rangle \le \left\langle c^{L}_{\lambda}, \bar{x} \right\rangle \quad \text{and} \quad \left\langle c^{R}_{\lambda}, x^* \right\rangle \le \left\langle c^{R}_{\lambda}, \bar{x} \right\rangle$$

Hence, for $\alpha = \alpha_0 \le \alpha_1 \le \cdots \le \alpha_{n-1} \le \alpha_n = 1$, we have

$$\left(\left\langle c^{L}_{\alpha_i}, x^* \right\rangle, \left\langle c^{R}_{\alpha_i}, x^* \right\rangle\right)^T \geqq \left(\left\langle c^{L}_{\alpha_i}, x \right\rangle, \left\langle c^{R}_{\alpha_i}, x \right\rangle\right)^T, \ i = 0, 1, \cdots, n$$

which contradicts the assumption that $x^* \in X_\alpha$ is a Pareto optimal solution to the MOLP$_\alpha$ problem.

THEOREM 20.

Let all the fuzzy parameters be piece-wise trapezoidal membership functions in FMOLP$_\alpha$:

$$\mu_{\tilde{z}}(t)=\begin{cases} 0 & t<z^L_{\alpha 0} \\ \dfrac{\alpha_1-\alpha_0}{z^L_{\alpha 1}-z^L_{\alpha 0}}\left(t-z^L_{\alpha 0}\right)+\alpha_0 & z^L_{\alpha 0}\leqq t<z^L_{\alpha 1} \\ \dfrac{\alpha_2-\alpha_1}{z^L_{\alpha 2}-z^L_{\alpha 1}}\left(t-z^L_{\alpha 1}\right)+\alpha_1 & z^L_{\alpha 1}\leqq t<z^L_{\alpha 2} \\ \cdots & \cdots \\ 1 & z^L_{\alpha n}\leqq t<z^R_{\alpha n} \\ \dfrac{\alpha_n-\alpha_{n-1}}{z^R_{\alpha n-1}-z^R_{\alpha n}}\left(-t+z^R_{\alpha n-1}\right)+\alpha_{n-1} & z^R_{\alpha n}\leqq t<z^R_{\alpha n-1} \\ \cdots & \cdots \\ \dfrac{\alpha_0-\alpha_1}{z^R_{\alpha 1}-z^R_{\alpha 0}}\left(-t+z^R_{\alpha 0}\right)+\alpha_0 & z^R_{\alpha 1}\leqq t\leqq z^R_{\alpha 0} \\ 0 & z^R_{\alpha 0}<t \end{cases} \tag{23}$$

and a point $x^* \in X$ be a feasible solution to the FMOLP problem. Then x^* is a weak Pareto optimal solution to the problem if and only if x^* is a weak Pareto optimal solution to the MOLP$_\alpha$ problem:

$$(\text{MOLP}_\alpha)\quad \begin{cases} \text{Maximize} & \langle c^L_{\alpha 0}, x\rangle \\ & \langle c^R_{\alpha 0}, x\rangle \\ & \langle c^L_{\alpha 1}, x\rangle \\ & \vdots \\ & \langle c^R_{\alpha n-1}, x\rangle \\ & \langle c^L_{\alpha n}, x\rangle \\ & \langle c^R_{\alpha n}, x\rangle \\ \text{subject to} & \sum_{j=1}^n a^L_{ij\alpha 0} x_j \leqq b^L_{i\alpha 0} \\ & \sum_{j=1}^n a^R_{ij\alpha 0} x_j \leqq b^R_{i\alpha 0} \\ & \sum_{j=1}^n a^L_{ij\alpha 1} x_j \leqq b^L_{i\alpha 1} \\ & \vdots \\ & \sum_{j=1}^n a^R_{ij\alpha 1} x_j \leqq b^R_{i\alpha 1} \\ & \sum_{j=1}^n a^L_{ij\alpha n} x_j \leqq b^L_{i\alpha n} \\ & \sum_{j=1}^n a^R_{ij\alpha n} x_j \leqq b^R_{i\alpha n} \\ & x_i \geqq 0 \end{cases} \tag{24}$$

Proof.
See Theorem 19.

Therefore, if we use existing methods to get a complete optimal solution x^* to the MOLP$_\alpha$ problem, then x^* is a complete optimal solution to the FMOLP$_\alpha$ problem. This gives a way to solve the FMOLP$_\alpha$ problems, which will be used in developing detailed FMOLP algorithms and methods.

4. FUZZY-GOAL MULTI-OBJECTIVE DECISION-MAKING MODEL

Decision makers may want to specify their fuzzy goals for the objective functions in dealing with the FMOLP problem (21) under some circumstances. The key idea behind goal programming is to get the optimal solution that has the minimized deviations from goals set by decision makers. In standard goal programming, goals need to be given by precise data. In practice, it is often difficult for a decision maker to provide a precise attainment for each objective function. Applying fuzzy set theory into goal programming makes it possible for decision makers to indicate their vague aspirations, which can be qualified by linguistic terms. Such goals can be expressed as, for instance, "possibly greater than g_1," "around g_2" or "substantially less than g_3." These types of linguistic terms can then be qualified by eliciting membership functions of fuzzy sets.

Considering the FMOLP$_\alpha$ problem for the fuzzy multiple objective functions $\langle \tilde{c}, x \rangle_F$, any decision maker can specify fuzzy goals $\tilde{g} = (\tilde{g}_1, \tilde{g}_2, \ldots, \tilde{g}_k)^T$ under a satisfactory degree α that reflects the desired values of the objective functions of the decision maker. These fuzzy goals can be represented by fuzzy numbers with any form of membership functions. By defining a fuzzy deviation function $\tilde{D}(\langle \tilde{c}, x \rangle_F, \tilde{g})$ as a fuzzy difference between the fuzzy objective function $\langle \tilde{c}, x \rangle_F$ and fuzzy goals $\tilde{g} = (\tilde{g}_1, \tilde{g}_2, \ldots, \tilde{g}_k)^T$, the FMOGP$_\alpha$ problem under a satisfactory degree α is formulated as follows:

$$(\text{FMOGP}_\alpha)\begin{cases}\text{Minimize}_\alpha \tilde{D}\left(\langle \tilde{c},\ x\rangle_F, \tilde{g}\right) \\ \text{subject to } \tilde{A}x \underset{=\alpha}{\prec} \tilde{b} \\ x \geqq 0\end{cases} \tag{25}$$

that is, find an $x^* \in \tilde{X}_\alpha$, which minimizes $\tilde{D}\left(\langle \tilde{c}, x\rangle_F, \tilde{g}\right)$ or

$$x^* = \arg \min_{x\in X}{}_\alpha \tilde{D}\left(\langle \tilde{c}, x\rangle_F, \tilde{g}\right). \tag{26}$$

Normally, the fuzzy distance function $\tilde{D}\left(\langle \tilde{c}, x\rangle_F, \tilde{g}\right)$ is defined as a maximum of deviations of individual goals,

$$\tilde{D}\left(\langle \tilde{c}, x\rangle_F, \tilde{g}\right) = \max_{i=1,\ldots,k}\left\{\tilde{D}_i\left(\sum_{j=1}^{n} \tilde{c}_{ij}x_j, \tilde{g}_i\right)\right\}. \tag{27}$$

By Eq. (26), the FMOGP_α problem (25) is converted as follows:

$$\begin{aligned} &\text{Min} \max_{i=1,\ldots,k}\left\{\tilde{D}_i\left(\sum_{j=1}^{n} \tilde{c}_{ij}x_j, \tilde{g}_i\right)\right\} \\ &\text{subject to } \tilde{A}x \underset{=\alpha}{\prec} \tilde{b} \\ &\qquad x \geqq 0 \end{aligned} \tag{28}$$

where

$$\begin{aligned} \tilde{D}_i\left(\sum_{j=1}^{n} \tilde{c}_{ij}x_j, \tilde{g}_i\right) &= \max_{\lambda\in[0,1]}\left\{\left|\sum_{j=1}^{n} \tilde{c}_{ij\lambda}^{L}x_j - g_{i\lambda}^{L}\right|, \left|\sum_{j=1}^{n} \tilde{c}_{ij\lambda}^{R}x_j - g_{i\lambda}^{R}\right|\right\} \\ &= \max_{\lambda\in[0,1]}\left\{\left|c_{i\lambda}^{L}x - g_{i\lambda}^{L}\right|, \left|c_{i\lambda}^{R}x - g_{i\lambda}^{R}\right|\right\} \qquad i = 1,\ldots,k. \end{aligned} \tag{29}$$

From Eq. (29), the optimal solution of Eq. (28) can be obtained by solving the following GP model:

$$
(\mathrm{GP}_{\alpha\lambda\text{-}1})\begin{cases}\min \max\limits_{\substack{i=1,\ldots,k\\ \lambda\in[\alpha,1]}} \left\{ c_{i\lambda}^{L}x - g_{i\lambda}^{L}, c_{i\lambda}^{R}x - g_{i\lambda}^{R} \right\} \\ \\ \text{subject to } \; A_{\lambda}^{L}x \le b_{\lambda}^{L},\; A_{\lambda}^{R}x \le b_{\lambda}^{R},\; \forall\lambda\in[\alpha,1] \\ \qquad\qquad\quad x \geqq 0 \end{cases} \tag{30}
$$

or

$$
(\mathrm{GP}_{\alpha\lambda\text{-}2})\begin{cases}\min \max\limits_{\substack{i=1,\ldots,k\\ \lambda\in[\alpha,1]}} \left\{ g_{i\lambda}^{L} - c_{i\lambda}^{L}x, g_{i\lambda}^{R} - c_{i\lambda}^{R}x \right\} \\ \\ \text{subject to } \; A_{\lambda}^{L}x \le b_{\lambda}^{L},\; A_{\lambda}^{R}x \le b_{\lambda}^{R},\; \forall\lambda\in[\alpha,1] \\ \quad x \geqq 0 \end{cases} \tag{31}
$$

where

$$
\begin{pmatrix} c_{1\lambda}^{L} \\ c_{2\lambda}^{L} \\ \vdots \\ c_{k\lambda}^{L} \end{pmatrix} = \begin{bmatrix} c_{11\lambda}^{L} & c_{12\lambda}^{L} & \cdots & c_{1n\lambda}^{L} \\ c_{21\lambda}^{L} & c_{22\lambda}^{L} & \cdots & c_{2n\lambda}^{L} \\ \vdots & \vdots & \ddots & \vdots \\ c_{k1\lambda}^{L} & c_{k2\lambda}^{L} & \cdots & c_{kn\lambda}^{L} \end{bmatrix}, \begin{pmatrix} c_{1\lambda}^{R} \\ c_{2\lambda}^{R} \\ \vdots \\ c_{k\lambda}^{R} \end{pmatrix} = \begin{bmatrix} c_{11\lambda}^{R} & c_{12\lambda}^{R} & \cdots & c_{1n\lambda}^{R} \\ c_{21\lambda}^{R} & c_{22\lambda}^{R} & \cdots & c_{2n\lambda}^{R} \\ \vdots & \vdots & \ddots & \vdots \\ c_{k1\lambda}^{R} & c_{k2\lambda}^{R} & \cdots & c_{kn\lambda}^{R} \end{bmatrix}
$$

The adoption of $\mathrm{GP}_{\alpha\lambda\text{-}1}$ (30) or $\mathrm{GP}_{\alpha\lambda\text{-}2}$ (31) for solving the FMOGP_{α} problem depends on the relationship of $\langle\tilde{c}, x\rangle_F$ and $\tilde{g}$; i.e., if $\langle\tilde{c}, x\rangle_F \succeq \tilde{g}$, then $\mathrm{GP}_{\alpha\lambda\text{-}1}$ (30) is used; otherwise, $\mathrm{GP}_{\alpha\lambda\text{-}2}$ (31) is adopted.

Hence, when we get a complete optimal solution x^* to the goal programming problem, x^* is a complete optimal solution to the FMOLP problem.

5. AN INTERACTIVE FUZZY-GOAL FUZZY MULTI-OBJECTIVE DECISION-MAKING METHOD

5.1 Fuzzy Goal-Based Interaction

Many decision makers prefer an interactive approach to finding an optimal solution for their decision problem as such an approach enables them to directly engage in the problem-solving process. This section proposes an interactive algorithm based on the fuzzy goal approximation algorithm. This algorithm not only allows decision makers to give their fuzzy goals but also allows them to continuously revise and adjust their fuzzy goals. Decision makers can then explore various optimal solutions under their goals and choose the most satisfactory one.

From the definitions of both FMOLP$_\alpha$ and MOLP$_{\alpha\lambda}$ problems, any decision maker can set up their fuzzy goals $\tilde{g} = (\tilde{g}_1, \tilde{g}_2, \ldots, \tilde{g}_k)^T$ under a satisfactory degree α. Its corresponded optimal solution, which results in the objective values being the nearest to the fuzzy goals, is obtained by solving the following minimax problem:

$$(\text{MOLP}\,\alpha) = \begin{cases} \min \ \max \begin{pmatrix} C_\lambda^L x - g_\lambda^L \\ C_\lambda^R x - g_\lambda^R \end{pmatrix}, \forall \lambda \in [\alpha, 1] \\ \text{subject to} \quad x \in X = \left\{ x \in R^n \mid A_\lambda^L x \le b_\lambda^L, A_\lambda^R x \le b_\lambda^R, x \ge 0, \forall \lambda \in [\alpha, 1] \right\} \end{cases} \tag{32}$$

where

$$g_\lambda^L = \left[g_{1\lambda}^L, g_{2\lambda}^L, \cdots, g_{k\lambda}^L \right]^T, \quad g_\lambda^R = \left[g_{1\lambda}^R, g_{2\lambda}^R, \cdots, g_{k\lambda}^R \right]^T$$

$$C_\lambda^L = \begin{bmatrix} c_{11\lambda}^L & c_{12\lambda}^L & \cdots & c_{1n\lambda}^L \\ c_{21\lambda}^L & c_{22\lambda}^L & \cdots & c_{2n\lambda}^L \\ \vdots & \vdots & \ddots & \vdots \\ c_{k1\lambda}^L & c_{k2\lambda}^L & \cdots & c_{kn\lambda}^L \end{bmatrix}, \quad C_\lambda^R = \begin{bmatrix} c_{11\lambda}^R & c_{12\lambda}^R & \cdots & c_{1n\lambda}^R \\ c_{21\lambda}^R & c_{22\lambda}^R & \cdots & c_{2n\lambda}^R \\ \vdots & \vdots & \ddots & \vdots \\ c_{k1\lambda}^R & c_{k2\lambda}^R & \cdots & c_{kn\lambda}^R \end{bmatrix}$$

$$A_\lambda^L = \begin{bmatrix} a_{11\lambda}^L & a_{12\lambda}^L & \cdots & a_{1n\lambda}^L \\ a_{21\lambda}^L & a_{22\lambda}^L & \cdots & a_{2n\lambda}^L \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1\lambda}^L & a_{m2\lambda}^L & \cdots & a_{mn\lambda}^L \end{bmatrix}, A_\lambda^R = \begin{bmatrix} a_{11\lambda}^R & a_{12\lambda}^R & \cdots & a_{1n\lambda}^R \\ a_{21\lambda}^R & a_{22\lambda}^R & \cdots & a_{2n\lambda}^R \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1\lambda}^R & a_{m2\lambda}^R & \cdots & a_{mn\lambda}^R \end{bmatrix} \tag{33}$$

$$b_\lambda^L = \left[b_{1\lambda}^L, b_{2\lambda}^L, \cdots, b_{m\lambda}^L\right]^T, \; b_\lambda^R = \left[b_{1\lambda}^R, b_{2\lambda}^R, \cdots, b_{m\lambda}^R\right]^T.$$

Let the interval $[\alpha, 1]$ be decomposed into l mean sub-intervals with $(l+1)$ nodes $\lambda_i \; (i = 0, \cdots, l)$ that are arranged in the order of

$$\alpha = \lambda_0 < \lambda_1 < \cdots < \lambda_l = 1.$$

Based on the current decompositions, we denote:

$$(\text{MOLP}_{\alpha\lambda m})_l \begin{cases} \min \max \begin{pmatrix} c_{i\lambda_j}^L x - g_{i\lambda_{ji}}^L \\ c_{i\lambda_j}^R x - g_{i\lambda_{ji}}^R \end{pmatrix}, & i = 1, 2, \cdots, k, \quad j = 1, 2, \cdots, l, \\ \text{subject to} \quad x \in X^l \end{cases} \tag{34}$$

where $X^l = \bigcap_i^l X_{\lambda i}$, $X_{\lambda i} = \left\{x \in R^n \mid A_{\lambda i}^L x \le b_{\lambda i}^L, A_{\lambda i}^R x \le b_{\lambda i}^R, x \ge 0\right\}, \lambda \in [\alpha, 1]$.

5.2 Description of the Algorithm

This algorithm consists of 11 steps within two stages. Stage 1 aims to find an initial optimal solution for the problem. Stage 2 is an interactive process in which when a decision maker specifies a set of fuzzy goals for related objective functions, an optimal solution is generated. By revising fuzzy goals, this algorithm will provide the decision maker with a series of optimal solutions from which the decision maker can select the most suitable one on the basis of preference, judgment, and experience.

The algorithm is described as follows:

Stage 1: Initialization

Step 1. Select an initial satisfactory degree α $\left(0 \le \alpha \le 1\right)$, give the membership function of $\tilde{c}$ for $\tilde{f}(x) = \tilde{c}x$, $\tilde{a}$ and $\tilde{b}$ for $\tilde{a}x \precapprox_{\alpha} \tilde{b}$, and set weights for objective functions by the decision maker.

Step 2. Set $l = 1$, then solve

$$(\mathrm{MOLP}_{\alpha\lambda})_l \begin{cases} \max & \begin{pmatrix} c^L_{i\lambda_j} x \\ c^R_{i\lambda_j} x \end{pmatrix}, \qquad i = 1, \cdots, k; \quad j = 0, 1, \cdots, l, \\ \text{subject to.} & x \in X^l \end{cases} \tag{35}$$

with the solution $\left(x\right)_l$, where $(x)_l = (x_1, x_2, \ldots, x_n)_l$, and the solution obtained is subject to the constraint $x \in X^l$.

Step 3. Solve $(\mathrm{MOLP}_{\alpha\lambda})_{2l}$ with the solution $(x)_{2l}$, subject to the constraint $x \in X^{2l}$.

The interval [α, 1] is further split. Suppose there are $\left(l+1\right)$ nodes λ_i $\left(i = 0,\ 2,\ 4, \ldots,\ 2l\right)$ in the interval [α, 1], and l new nodes λ_i $\left(i = 1,\ 3,\ \ldots,\ 2l-1\right)$ are inserted. The relationship between the new nodes and previous ones is:

$$\lambda_{2i+1} = \frac{\lambda_{2i} + \lambda_{2i+2}}{2}, \quad i = 0,\ 1,\ \ldots,\ l-1. \tag{36}$$

Therefore, each of the fuzzy objective functions is converted into $2 \times \left(2l+1\right)$ non-fuzzy objective functions, and the same conversion happens for the constraints $\tilde{a}_i x \precapprox_{\alpha} \tilde{b}_i$. The solution $\left(x\right)_{2l}$ is now based on the set of updated (including original) nonfuzzy objective functions and nonfuzzy constraints.

Step 4. If $\left\| \left(x\right)_{2l} - \left(x\right)_l \right\| < \varepsilon$, then $(x)_{2l}$ is the final solution of the $\mathrm{MOLP}_{\alpha\lambda}$ problem. Otherwise, update l to $2l$ and go back to Step 3.

Step 5. If the corresponded Pareto optimal solution x^* exists, go forward to Step 6. Otherwise, the decision maker must go back to Step 1 to reassign a degree α (give a higher value for the degree α).

Step 6. If the decision maker is satisfied with the Pareto optimal solution, the interactive process terminates. Otherwise, go to Stage 2.

Stage 2: Iteration

As the decision maker is not satisfied with the obtained optimal solution in the *Initialization* stage (or the previous iteration phase), the decision maker specifies fuzzy goals (or revised current goals) for the fuzzy objective functions. A new compromise solution is then generated. This process will terminate when the decision maker finds a satisfactory solution.

Step 7. Give a set of new fuzzy goals or revise current fuzzy goals according to the decision maker. At the same time, a satisfactory degree α can be revised as well. The original decision problem is therefore covered into an $(\text{MOLP}_{\alpha\lambda m})_l$ problem.

Step 8. Set $l = 1$; solve $(\text{MOLP}_{\alpha\lambda m})_l$ with the solution $(x)_l$, subject to the constraint $x \in X^l$.

Let $\lambda_0 = \alpha$ and $\lambda_1 = 1$ in the interval $[\alpha, 1]$; each fuzzy objective function $\tilde{f}_i(x) = \tilde{c}_i x$ under the fuzzy goal $\tilde{g} = (\tilde{g}_1, \tilde{g}_2, \ldots, \tilde{g}_k)^T$, and related constraints are converted into non-fuzzy forms.

Step 9. Solve $(\text{MOLP}_{\alpha\lambda m})_{2l}$ with the solution $(x)_{2l}$ subject to the constraint $x \in X^{2l}$.

Similar to Step 6, the interval $[\alpha, 1]$ is further split, and new nodes are inserted further. Fuzzy objective functions under related fuzzy goals and constraints are converted into non fuzzy again. A new solution $(x)_{2l}$ is generated.

Step 10. If $\|(x)_{2l} - (x)_l\| < \varepsilon$, then $(x)_{2l}$ is the final solution of the $\text{MOLP}_{\alpha\lambda m}$ problem. Otherwise, update l to $2l$ and go back to Step 9.

Step 11. If the decision maker is satisfied with the current Pareto optimal solution obtained in Step 10, the interactive process terminates, and the current optimal solution is the final satisfactory solution to the decision maker. Otherwise, go back to Step 7.

We now give another explanation for this algorithm:

Definition 1 is about ranking two n-dimensional fuzzy numbers under a satisfactory degree α. This definition is the foundation for the comparison of fuzzy objective functions and the left- and right-hand sides of fuzzy constraints in an FMOLP problem. In Step 5 of this method, if the Pareto optimal solution does not exist under a satisfactory degree α, replacing this α with a higher value may derive a Pareto optimal solution.

In Step 7 of the algorithm, the decision maker can improve goals for some unsatisfactory objectives by sacrificing the goals of others. The new fuzzy goals can be given directly by a new fuzzy number vector or by increasing/decreasing the values of its corresponded objective functions in a current Pareto optimal solution.

Figure 2 shows the flowchart of the fuzzy goal interactive algorithm.

6. A NUMERAL EXAMPLE

To illustrate the interactive fuzzy-goal multi-objective algorithm, we consider the following FMOLP$_\alpha$ problem with two fuzzy objective functions and four fuzzy constraints:

$$\max \tilde{f}(x) = \max \begin{pmatrix} \tilde{f}_1(x) \\ \tilde{f}_2(x) \end{pmatrix} = \max \begin{pmatrix} \tilde{c}_{11}x_1 + \tilde{c}_{12}x_2 \\ \tilde{c}_{21}x_1 + \tilde{c}_{22}x_2 \end{pmatrix} = \max \begin{pmatrix} \tilde{4}x_1 + \tilde{2}x_2 \\ -\tilde{2}x_1 + \tilde{4}x_2 \end{pmatrix}$$

$$\text{subject to} \quad \begin{cases} \tilde{a}_{11}x_1 + \tilde{a}_{12}x_2 = -\tilde{1}x_1 + \tilde{3}x_2 \preceq \tilde{b}_1 = \tilde{21} \\ \tilde{a}_{21}x_1 + \tilde{a}_{22}x_2 = \tilde{1}x_1 + \tilde{3}x_2 \preceq \tilde{b}_2 = \tilde{27} \\ \tilde{a}_{31}x_1 + \tilde{a}_{32}x_2 = \tilde{4}x_1 + \tilde{3}x_2 \preceq \tilde{b}_3 = \tilde{45} \\ \tilde{a}_{41}x_1 + \tilde{a}_{42}x_2 = \tilde{3}x_1 + \tilde{1}x_2 \preceq \tilde{b}_4 = \tilde{30} \\ x_1 \leq 0; \quad x_2 \leq 0 \end{cases}$$

The membership functions of the parameters of the objective functions and constraints are set up as follows:

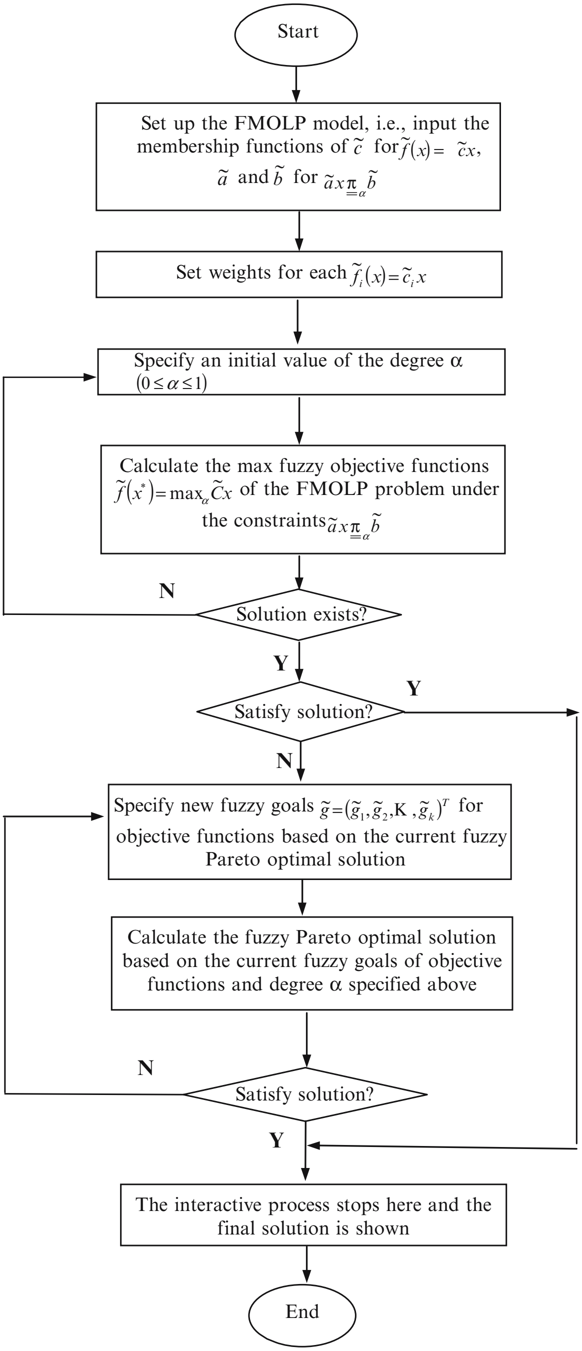

Figure 2. Flow chart for the fuzzy−goal interactive algorithm

$$\mu_{\tilde{c}_{11}}(x)=\begin{cases}0 & x<3 \text{ or } 6<x\\ (x^2-9)/7 & 3\le x<4\\ 1 & x=4\\ (36-x^2)/20 & 4<x\le 6\end{cases}\qquad \mu_{\tilde{c}_{12}}(x)=\begin{cases}0 & x<1 \text{ or } 4<x\\ (x^2-1)/3 & 1\le x<2\\ 1 & x=2\\ (16-x^2)/12 & 2<x\le 4\end{cases}$$

$$\mu_{\tilde{c}_{21}}(x)=\begin{cases}0 & x<-2.5 \text{ or } -1<x\\ (6.25-x^2)/2.25 & -2.5\le x<-2\\ 1 & x=-2\\ (x^2-1)/3 & -2<x\le -1\end{cases}\qquad \mu_{\tilde{c}_{22}}(x)=\begin{cases}0 & x<3 \text{ or } 6<x\\ (x^2-9)/7 & 3\le x<4\\ 1 & x=4\\ (36-x^2)/20 & 4<x\le 6\end{cases}$$

$$\mu_{\tilde{a}_{11}}(x)=\begin{cases}0 & x<-2 \text{ or } -0.5<x\\ (4-x^2)/3 & -2\le x<-1\\ 1 & x=-1\\ (x^2-0.25)/0.75 & -1<x\le -0.5\end{cases}\qquad \mu_{\tilde{a}_{12}}(x)=\begin{cases}0 & x<2 \text{ or } 5<x\\ (x^2-4)/5 & 2\le x<3\\ 1 & x=3\\ (25-x^2)/16 & 3<x\le 5\end{cases}$$

$$\mu_{\tilde{a}_{21}}(x)=\begin{cases}0 & x<0.5 \text{ or } 2<x\\ (x^2-0.25)/0.75 & 0.5\le x<1\\ 1 & x=1\\ (4-x^2)/3 & 1<x\le 2\end{cases}\qquad \mu_{\tilde{a}_{22}}(x)=\begin{cases}0 & x<2 \text{ or } 5<x\\ (x^2-4)/5 & 2\le x<3\\ 1 & x=3\\ (25-x^2)/16 & 3<x\le 5\end{cases}$$

$$\mu_{\tilde{c}_{31}}(x)=\begin{cases}0 & x<3 \text{ or } 6<x\\ (x^2-9)/7 & 3\le x<4\\ 1 & x=4\\ (36-x^2)/20 & 4<x\le 6\end{cases}\qquad \mu_{\tilde{a}_{32}}(x)=\begin{cases}0 & x<2 \text{ or } 5<x\\ (x^2-4)/5 & 2\le x<3\\ 1 & x=3\\ (25-x^2)/16 & 3<x\le 5\end{cases}$$

$$\mu_{\tilde{a}_{41}}(x)=\begin{cases}0 & x<2 \text{ or } 5<x\\ (x^2-4)/5 & 2\le x<3\\ 1 & x=3\\ (25-x^2)/16 & 3<x\le 5\end{cases}\qquad \mu_{\tilde{a}_{42}}(x)=\begin{cases}0 & x<0.5 \text{ or } 2<x\\ (x^2-0.25)/0.75 & 0.5\le x<1\\ 1 & x=1\\ (4-x^2)/3 & 1<x\le 2\end{cases}$$

$$\mu_{\tilde{b}_1}(x)=\begin{cases}0 & x<20 \text{ or } 23<x\\ (x^2-400)/41 & 20\le x<21\\ 1 & x=21\\ (529-x^3)/88 & 21<x\le 23\end{cases}\qquad \mu_{\tilde{b}_2}(x)=\begin{cases}0 & x<26 \text{ or } 29<x\\ (x^2-676)/53 & 26\le x<27\\ 1 & x=27\\ (841-x^2)/112 & 27<x\le 29\end{cases}$$

$$\mu_{\tilde{b}_3}(x)=\begin{cases}0 & x<44 \text{ or } 47<x\\ (x^2-1936)/89 & 44\le x<45\\ 1 & x=.45\\ (2209-x^2)/184 & 45<x\le 47\end{cases}\qquad \mu_{\tilde{b}_4}(x)=\begin{cases}0 & x<29 \text{ or } 32<x\\ (x^2-841)/59 & 29\le x<30\\ 1 & x=.30\\ (1024-x^2)/124 & 30<x\le 32\end{cases}$$

Stage 1: Initialization

Step 1. Input membership functions of $\tilde{c}$ for objective functions $\tilde{f}(x)=\tilde{c}x$, $\tilde{a}$ and $\tilde{b}$ for constraints $\tilde{a}x \underset{=\alpha}{\prec} \tilde{b}$. We set an initial satisfactory degree α as 0.2. We use default values for the weights of objective functions.

Steps 2–4. Under the degree α = 0.2, we calculate the Pareto optimal solution. Associated with the FMOLP$_\alpha$ problem in this example, a corresponding MOLP$_{\alpha\lambda}$ problem is listed:

$$\max \begin{bmatrix} \sqrt{9\lambda+9} & \sqrt{3\lambda+1} \\ \sqrt{36-20\lambda} & \sqrt{16-12\lambda} \\ \sqrt{6.25-2.25\lambda} & \sqrt{9\lambda+9} \\ \sqrt{3\lambda+1} & \sqrt{36-20\lambda} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

subject to

$$\begin{bmatrix} \sqrt{4-3\lambda} & \sqrt{5\lambda+4} \\ \sqrt{0.75\lambda+0.25} & \sqrt{25-16\lambda} \\ \sqrt{0.75\lambda+0.25} & \sqrt{5\lambda+4} \\ \sqrt{4-3\lambda} & \sqrt{25-16\lambda} \\ \sqrt{9\lambda+9} & \sqrt{5\lambda+4} \\ \sqrt{36-20\lambda} & \sqrt{25-16\lambda} \\ \sqrt{5\lambda+4} & \sqrt{0.75\lambda+0.25} \\ \sqrt{25-16\lambda} & \sqrt{4-3\lambda} \end{bmatrix} \begin{bmatrix} x_1 \\ x_1 \end{bmatrix} \le \begin{bmatrix} \sqrt{41\lambda+400} \\ \sqrt{529-88\lambda} \\ \sqrt{53\lambda+676} \\ \sqrt{841-112\lambda} \\ \sqrt{89\lambda+1936} \\ \sqrt{2209-184\lambda} \\ \sqrt{59\lambda+841} \\ \sqrt{1024-124\lambda} \end{bmatrix}$$

where $\forall\lambda \in [\alpha, 1]$.

Refer to the MOLP$_{\alpha\lambda}$ problem, initially $\lambda_0 = 0.2$ and $\lambda_1 = 1$; then 8 non fuzzy objective functions and 16 non fuzzy constraints are generated. The result is listed as follows:

$$\max \begin{bmatrix} \sqrt{10.8} & \sqrt{1.6} \\ \sqrt{18} & 2 \\ \sqrt{32} & \sqrt{13.6} \\ 4 & 2 \\ \sqrt{5.8} & \sqrt{10.8} \\ 2 & \sqrt{18} \\ \sqrt{1.6} & \sqrt{32} \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

subject to

$$\begin{bmatrix} \sqrt{3.4} & \sqrt{5} \\ 1 & 3 \\ \sqrt{0.4} & \sqrt{21.8} \\ 1 & 3 \\ \sqrt{0.4} & \sqrt{5} \\ 1 & 3 \\ \sqrt{3.4} & \sqrt{21.8} \\ 1 & 3 \\ \sqrt{10.8} & \sqrt{5} \\ \sqrt{18} & 3 \\ \sqrt{32} & \sqrt{21.8} \\ 4 & 3 \\ \sqrt{5} & \sqrt{0.4} \\ 3 & 1 \\ \sqrt{21.8} & \sqrt{3.4} \\ 3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_1 \end{bmatrix} \le \begin{bmatrix} \sqrt{408.2} \\ 21 \\ \sqrt{501.6} \\ 21 \\ \sqrt{686.6} \\ 27 \\ \sqrt{818.6} \\ 27 \\ \sqrt{1953.8} \\ 45 \\ \sqrt{2245.8} \\ 45 \\ \sqrt{852.8} \\ 30 \\ \sqrt{999.2} \\ 30 \end{bmatrix}$$

The interval $[\alpha, 1]$ is further split. We then have

$$x_1^* = 1.9115$$
$$x_2^* = 5.1023$$

and two optimal objective values

$$\tilde{f}_1^*\left(x_1^*, x_2^*\right) = \tilde{f}_1^*\left(1.9115, 5.1023\right) = 1.9115\tilde{c}_{11} + 5.1023\tilde{c}_{12}$$

$$\tilde{f}_2^*\left(x_1^*, x_2^*\right) = \tilde{f}_2^*\left(1.9115, 5.1023\right) = 1.9115\tilde{c}_{21} + 5.1023\tilde{c}_{22}.$$

Steps 5 and 6. Suppose the decision maker is not satisfied with the initial Pareto optimal solution; the interactive process will start.

Stage 2: Iterations
Iteration No. 1:

Step 7. Based on the Pareto optimal solution obtained in Stage 1, the decision maker specifies new fuzzy goals $(\tilde{g}_1, \tilde{g}_2)$ by increasing 30% of the value of the first objective function $\tilde{f}_1^*(x_1^*, x_2^*) = \tilde{f}_1^*(1.9115, 5.1023) = 1.9115\tilde{c}_{11} + 5.1023\tilde{c}_{12}$ and decreasing 25% of the value of the second one $\tilde{f}_2^*(x_1^*, x_2^*) = \tilde{f}_2^*(1.9115, 5.1023) = 1.9115\tilde{c}_{21} + 5.1023\tilde{c}_{22}$. That is,

$$(\tilde{g}_1, \tilde{g}_2) = \left(1.3 * \tilde{f}_1^*(x_1^*, x_2^*),\ 0.75 * \tilde{f}_2^*(x_1^*, x_2^*)\right)$$

Steps 8–10. Calculate the fuzzy Pareto optimal solution based on the new fuzzy goals $(\tilde{g}_1, \tilde{g}_2)$ and the satisfactory degree $\alpha = 0.2$.

Under the new fuzzy goals, the FMOLP$_\alpha$ problem is converted into a nonfuzzy MOLP$_{\alpha\lambda m}$ problem as follows:

$$\min\ \max \begin{bmatrix} \sqrt{9\lambda+9} & \sqrt{3\lambda+1} \\ \sqrt{36-20\lambda} & \sqrt{16-12\lambda} \\ \sqrt{6.25-2.25\lambda} & \sqrt{9\lambda+9} \\ \sqrt{3\lambda+1} & \sqrt{36-20\lambda} \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - \begin{pmatrix} 2.4849\sqrt{9\lambda+9} + 6.6329\sqrt{3\lambda+1} \\ 2.4849\sqrt{36-20\lambda} + 6.6329\sqrt{16-12\lambda} \\ 1.5292\sqrt{6.25-2.25\lambda} + 4.0818\sqrt{9\lambda+9} \\ 1.5292\sqrt{3\lambda+1} + 4.0818\sqrt{36-20\lambda} \end{pmatrix}$$

subject to

$$\begin{bmatrix} \sqrt{4-3\lambda} & \sqrt{5\lambda+4} \\ \sqrt{0.75\lambda+0.25} & \sqrt{25-16\lambda} \\ \sqrt{0.75\lambda+0.25} & \sqrt{5\lambda+4} \\ \sqrt{4-3\lambda} & \sqrt{25-16\lambda} \\ \sqrt{9\lambda+9} & \sqrt{5\lambda+4} \\ \sqrt{36-20\lambda} & \sqrt{25-16\lambda} \\ \sqrt{5\lambda+4} & \sqrt{0.75\lambda+0.25} \\ \sqrt{25-16\lambda} & \sqrt{4-3\lambda} \end{bmatrix} \begin{bmatrix} x_1 \\ x_1 \end{bmatrix} \le \begin{bmatrix} \sqrt{41\lambda+400} \\ \sqrt{529-88\lambda} \\ \sqrt{53\lambda+676} \\ \sqrt{841-112\lambda} \\ \sqrt{89\lambda+1936} \\ \sqrt{2209-184\lambda} \\ \sqrt{59\lambda+841} \\ \sqrt{1024-124\lambda} \end{bmatrix}$$

where $\forall\lambda \in [\alpha, 1]$;

We obtain

$$x_1^* = 3.0486 \quad x_2^* = 4.9239\ ,$$

and two optimal fuzzy objective values are

$$\widetilde{f}_1^*(x_1^*, x_2^*) = \widetilde{f}_1^*(3.0486, 4.9239) = 3.0486\widetilde{c}_{11} + 4.9239\widetilde{c}_{12}$$

$$\widetilde{f}_2^*(x_1^*, x_2^*) = \widetilde{f}_2^*(3.0486, 4.9239) = 3.0486\widetilde{c}_{21} + 4.9239\widetilde{c}_{22} .$$

Comparing the two groups of objective values, we can find that the first fuzzy objective function has some improvement, and the second one has some decrement.

Step 11. Suppose the decision maker does not satisfy the fuzzy Pareto optimal solution, the interactive process will proceed; that is, start the second iteration.

Iteration No. 2:

Step 7. At this iteration, suppose the decision maker specifies new fuzzy goals $(\widetilde{g}_1, \widetilde{g}_2)$ by the corresponding membership functions as follows:

$$\mu_{\widetilde{g}1}(x) = \begin{cases} 0 & x < 14 \text{ or } 37 < x \\ (x^2 - 196)/245 & 14 \le x < 21 \\ 1 & x = 21 \\ (1369 - x^2)/928 & 21 < x \le 37 \end{cases}$$

$$\mu_{\widetilde{g}2}(x) = \begin{cases} 0 & x < 6.5 \text{ or } 25 < x \\ (x^2 - 42.25)/114 & 6.5 \le x < 12.5 \\ 1 & x = 12.5 \\ (625 - x^2)/468.75 & 12.5 < x \le 25 \end{cases}$$

Steps 8–10. Calculate the fuzzy Pareto optimal solution based on the new fuzzy goals $(\widetilde{g}_1, \widetilde{g}_2)$, and keep the degree $\alpha = 0.2$.

Under the fuzzy goals, the FMOLP$_\alpha$ problem is converted into the non fuzzy MOLP$_{\alpha\lambda m}$ problem as follows:

min max

$$\begin{bmatrix} \sqrt{9\lambda+9} & \sqrt{3\lambda+1} \\ \sqrt{36-20\lambda} & \sqrt{16-12\lambda} \\ \sqrt{6.25-2.25\lambda} & \sqrt{9\lambda+9} \\ \sqrt{3\lambda+1} & \sqrt{36-20\lambda} \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - \begin{pmatrix} \sqrt{245\lambda+196} \\ \sqrt{1369-928\lambda} \\ \sqrt{114\lambda+42.25} \\ \sqrt{625-468.75\lambda} \end{pmatrix}$$

subject to

$$\begin{bmatrix} \sqrt{4-3\lambda} & \sqrt{5\lambda+4} \\ \sqrt{0.75\lambda+0.25} & \sqrt{25-16\lambda} \\ \sqrt{0.75\lambda+0.25} & \sqrt{5\lambda+4} \\ \sqrt{4-3\lambda} & \sqrt{25-16\lambda} \\ \sqrt{9\lambda+9} & \sqrt{5\lambda+4} \\ \sqrt{36-20\lambda} & \sqrt{25-16\lambda} \\ \sqrt{5\lambda+4} & \sqrt{0.75\lambda+0.25} \\ \sqrt{25-16\lambda} & \sqrt{4-3\lambda} \end{bmatrix} \begin{bmatrix} x_1 \\ x_1 \end{bmatrix} \le \begin{bmatrix} \sqrt{41\lambda+400} \\ \sqrt{529-88\lambda} \\ \sqrt{53\lambda+676} \\ \sqrt{841-112\lambda} \\ \sqrt{89\lambda+1936} \\ \sqrt{2209-184\lambda} \\ \sqrt{59\lambda+841} \\ \sqrt{1024-124\lambda} \end{bmatrix}$$

where $\forall\lambda \in [\alpha, 1]$.

We have

$$x_1^* = 2.8992 \quad x_2^* = 4.9829$$

and two optimal objective values are

$$\tilde{f}_1^*\left(x_1^*, x_2^*\right) = \tilde{f}_1^*(2.8992, 4.9829) = 2.8992\tilde{c}_{11} + 4.9829\tilde{c}_{12}$$

$$\tilde{f}_2^*\left(x_1^*, x_2^*\right) = \tilde{f}_2^*(2.8992, 4.9829) = 2.8992\tilde{c}_{21} + 4.9829\tilde{c}_{22}\,.$$

Step 11. Now the decision maker is satisfied with the fuzzy Pareto optimal solution obtained in Step 10; the interactive process thus terminates. The current fuzzy Pareto optimal solution is the final satisfactory solution of the FMOLP problem to the decision maker as follows:

$$\begin{cases} x_1^* = 2.8992 \\ x_2^* = 4.9829 \end{cases}$$

$$\begin{cases} \tilde{f}_1^*\left(x_1^*, x_2^*\right) = \tilde{f}_1^*(2.8992, 4.9829) = 2.8992\tilde{c}_{11} + 4.9829\tilde{c}_{12} \\ \tilde{f}_2^*\left(x_1^*, x_2^*\right) = \tilde{f}_2^*(2.8992, 4.9829) = 2.8992\tilde{c}_{21} + 4.9829\tilde{c}_{22} \end{cases}.$$

This example illustrates the proposed fuzzy-goal fuzzy multi-objective decision-making method.

7. CONCLUSION

This chapter presented a set of models and an interactive method to describe and solve the FMOLP problems. In the proposed FMOLP models, fuzzy parameters can appear in both objective functions and constraints and can be described by any form of membership function. When only objective functions or only constraints include fuzzy parameters, the model is still as an FMOLP problem since a real number is a special case of a fuzzy number. Similarly, a goal of a decision maker with a real number is also a special case of a fuzzy−goal in the models. The proposed FMODM method extends MODM decision analysis functions from a crisp to an imprecise scope and improved existing FMODM methods. It allows decision makers to express their goals by any form of membership function. When decision makers do not have a clear idea to how of choose a suitable form of membership function, they can try different forms. This feature offers decision makers a much higher confidence in using the method to solve their practical problems.

A decision support system has been developed to apply the method to assist decision makers to solve realistic FMOLP problems. This system has been initially tested by a number of examples, and results are very positive for our research project supported by the Australian Research Council (ARC).

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FUZZY OPTIMIZATION VIA MULTI-OBJECTIVE EVOLUTIONARY COMPUTATION FOR CHOCOLATE MANUFACTURING

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Abstract: This chapter outlines, first, a real-world industrial problem for product mix selection involving **8** variables and **21** constraints with fuzzy coefficients and, second, a multi-objective optimization approach to solve the problem. This problem occurs in production planning in which a decision maker plays a pivotal role in making decisions under a fuzzy environment. Decision maker should be aware of his/her level-of-satisfaction as well as degree of fuzziness while making the product mix decision. Thus, the authors have analyzed using a modified S-curve membership function for the fuzziness patterns and fuzzy sensitivity of the solution found from the multi-objective optimization methodology. An ad hoc Pareto-based multi-objective evolutionary algorithm is proposed to capture multiple nondominated solutions in a single run of the algorithm. Results obtained have been compared with the well-known multi-objective evolutionary algorithm NSGA-II.

Key words: Multi-objective optimization, evolutionary algorithm, NSGA-II

1. INTRODUCTION

It is well known that optimization problems originate in a variety of situations. Particularly interesting are those concerning management problems as decision makers usually state their data in a vague way: "high benefits," "as low as possible," "important savings," etc. Because of this vagueness, managers prefer to have not just one solution but a set of them, so that the most suitable solution can be applied according to the state of existing decision of the production process at a given time and without increasing delay. In these situations, fuzzy optimization is an ideal methodology, since it allows us to represent the underlying uncertainty of the optimization problem, while finding optimal solutions that reflect such uncertainty and then applying them to possible instances, once the uncertainty has been solved. This allows us to obtain a model of the behavior of the solutions based on the uncertainty of the optimization problem.

Fuzzy constrained optimization problems have been extensively studied since the 1970s. In the linear case, the first approaches to solve the so−called fuzzy linear programming problem appeared in Bellmann and Zadeh (1970), Tanaka et al. (1974), and in Zimmerman (1976). Since then, important contributions solving different linear models have been made and these models have been the subject of a substantial amount of work. In the nonlinear case (Ali, 1998; Ekel et al., 1998; Ramik and Vlach, 2002) the situation is quite different, as there is a wide variety of specific and both practically and theoretically relevant nonlinear problems, with each having a different solution method.

In this chapter a real-life industrial problem for product mix selection involving 21 constraints and 8 variables has been considered. This problem occurs in production planning in which a decision maker plays a pivotal role in making decisions under a highly fuzzy environment. Decision maker should be aware of his/her level-of-satisfaction as well as degree of fuzziness while making the product mix decision. Thus, the authors have analyzed using the sigmoidal membership function, the fuzziness patterns and fuzzy sensitivity of the solution. In Vasant (2003, 2004, 2006) a linear case of the problem is solved by using a linear programming iterative method that is repeatedly applied for different degrees of satisfaction values. In this chapter, a nonlinear case of the problem is considered and we propose a multi-objective optimization approach in order to capture solutions for different degrees of satisfaction with a simple run of the algorithm. This multi-objective optimization approach has been proposed by Jiménez et al. (2004a, 2004b, 2006) within a fuzzy optimization general context.

Given this background, this chapter is organized as follows: In section 2 a nonlinear case study in a chocolate manufacturing firm is described, and its mathematical formulation is stated. Section 3 we propose a multi-objective optimization approach for this problem and an ad hoc multi-objective evolutionary algorithm. Section 4 shows results obtained with the proposed multi-objective evolutionary algorithms and the well-known NSGA-II algorithm. Finally, Section 5 offers the main conclusions and future research.

2. NONLINEAR CASE STUDY IN A CHOCOLATE MANUFACTURING FIRM

Due to limitations in resources for manufacturing a product and the need to satisfy certain conditions in manufacturing and demand, a problem of fuzziness occurs in industrial systems. This problem occurs also in chocolate manufacturing when deciding a mixed selection of raw materials to produce varieties of products. This is referred here to as the product mix selection problem (Tabucanon, 1996).

There are a number of products to be manufactured by mixing different raw materials and using several varieties of processing. There are limitations in resources of raw materials and facility usage for the varieties of processing. The raw materials and facilities usage required for manufacturing each product are expressed by means of fuzzy coefficients. There are also some constraints imposed by the marketing department such as product mix requirement, main product line requirement, and lower and upper limit of demand for each product. It is necessary to obtain maximum profit with a certain degree of satisfaction of the decision maker.

2.1 Fuzzy Constrained Optimization Problem

The firm Chocoman Inc. manufactures eight different kinds of chocolate products. Input variables x_i represent the amount of manufacturated product in 10^3 units.

The function to maximize is the total profit obtained calculated as the summation of profit obtained with each product and taken into account the applied discount. Table 1 shows the profit (c_i) and discount (d_i) for each product i.

Table 1. Profit (c_i) and Discount (d_i) in \$ per 10^3 units

Product (xi)	Synonym	Profit (ci)	Discount (di)
x_1 = Milk chocolate, 250 g	MC 250	$c_1 = 180$	$d_1 = 0.18$
x_2 = Milk chocolate, 100 g	MC 100	$c_2 = 83$	$d_2 = 0.05$
x_3 = Crunchy chocolate, 250 g	CC 250	$c_3 = 153$	$d_3 = 0.15$
x_4 = Crunchy chocolate, 100 g	CC 100	$c_4 = 72$	$d_4 = 0.06$
x_5 = Chocolate with nuts, 250 g	CN 250	$c_5 = 130$	$d_5 = 0.13$
x_6 = Chocolate with nuts, 100 g	CN 100	$c_6 = 70$	$d_6 = 0.14$
x_7 = Chocolate candy	CANDY	$c_7 = 208$	$d_7 = 0.21$
x_8 = Chocolate wafer	WAFER	$c_8 = 83$	$d_8 = 0.1$

The lower limit of demand for each product i is 0 in all cases, whereas the upper limit (u^i) is shown in Table 2.

Table 2. Demand (u_i) in $ per 103 Units

Product	Demand (ui)
MC 250	$u_1 = 500$
MC 100	$u_2 = 800$
CC 250	$u_3 = 400$
CC 100	$u_4 = 600$
CN 250	$u_5 = 300$
CN 100	$u_6 = 500$
CANDY	$u_7 = 200$
WAFER	$u_8 = 400$

There are eight raw materials to be mixed in different proportions and nine processes (facilities) to be utilized. Therefore, there are 17 constraints with fuzzy coefficients separated in two sets such as raw material availability and facility capacity. These constraints are inevitable for each material and facility that is based on the material consumption, facility usage, and the resource availability. Table 3 shows fuzzy coefficients $\tilde{a}_{ij}$ represented by $\left(a_{ij}^l, a_{ij}^h\right)$ for required materials and facility usage j for manufacturing each product i and nonfuzzy coefficients b_j for availability of material or facility j.

Table 3. Raw Material and Facility Usage Required (per 10^3 units) $\left(a_{ij}^l, a_{ij}^h\right)$ and Availability (b_j)

	MC 250	MC 100	CC 250	CC 100	CN 250	CN 100	CANDY	WAFER	Availability
1	66, 109	26, 44	56, 94	22, 37	37, 62	15, 25	45, 75	9, 21	100,000
2	47, 78	19, 31	37, 62	15, 25	37, 62	15, 25	22, 37	9, 21	120,000
3	0, 0	0, 0	28, 47	11, 19	56, 94	22, 37	0, 0	0, 0	60,000
4	75, 125	30, 50	66, 109	26, 44	56, 94	22, 37	157, 262	18, 30	200,000
5	0, 0	0, 0	0, 0	0, 0	0, 0	0, 0	0, 0	54, 90	20,000
6	375, 625	0, 0	375, 625	0, 0	0, 0	0, 0	0, 0	187, 312	500,000
7	337, 562	0, 0	337, 563	0, 0	337, 562	0, 0	0, 0	0, 0	500,000
8	45, 75	95, 150	45, 75	90, 150	45, 75	90, 150	1200, 2000	187, 312	500,000
9	0.4, 0.6	0.1, 0.2	0.3, 0.5	0.1, 0.2	0.3, 0.4	0.1, 0.2	0.4, 0.7	0.1, 0.12	1000
10	0, 0	0, 0	0.1, 0.2	0.04, 0.07	0.2, 0.3	0.07, 0.12	0, 0	0, 0	200
11	0.6, 0.9	0.2, 0.4	0.6, 0.9	0.2, 0.4	0.6, 0.9	0.2, 0.4	0.7, 38718	0.3, 0.4	1500
12	0, 0	0, 0	0.2, 0.3	0.07, 0.12	0, 0	0, 0	0, 0	0, 0	200
13	0, 0	0, 0	0, 0	0, 0	0, 0	0, 0	0, 0	0.2, 0.4	100
14	0.07, 0.12	0.07, 0.12	0.07, 0.12	0.07, 0.12	0.07, 0.12	0.07, 0.12	0.15, 0.25	0, 0	400
15	0.2, 0.3	0, 0	0.2, 0.3	0, 0	0.2, 0.3	0, 0	0, 0	0, 0	400
16	0.04, 0.06	0.2, 0.4	0.04, 0.06	0.2, 0.4	0.04, 0.06	0.2, 0.4	1.9, 3.1	0.1, 0.2	1200
17	0.2, 0.4	0.2, 0.4	0.2, 0.4	0.2, 0.4	0.2, 0.4	0.2, 0.4	1.9, 3.1	1.9, 3.1	1000

Material or Facility

Cocoa (kg), Milk (kg), Nuts (kg), Cons.sugar (kg), Flour (kg), Alum.foil (ft^2), Paper(ft^2), Plastic (ft^2), Cooking(ton-hours), Mixing(ton-hours), Forming(ton-hours), Grinding(ton-hours), Wafer making(ton-hours), Cutting(hours), Packaging 1(hours), Packaging 2(hours), Labor(hours)

Additionally, the following constraints were established by the sales department of Chocoman Inc.:

1. Main product line requirement. The total sales from candy and wafer products should not exceed 15% of the total revenues from the chocolate bar products. Table 4 shows the values of sales/revenues (*ri*) for each product *i*.
2. Product mix requirements. Large-sized products (250 g) of each type should not exceed 60% of the small-sized product (100 g).

Table 4. Revenues/Sales (r_i) in $ per 103 Units

Product	Revenues/Sales (ri)
MC 250	$r_1 = 375$
MC 100	$r_2 = 150$
CC 250	$r_3 = 400$
CC 100	$r_4 = 160$
CN 250	$r_5 = 420$
CN 100	$r_6 = 175$
CANDY	$r_7 = 400$
WAFER	$r_8 = 150$

2.2 Membership Function for Coefficients

We consider the modified S-curve membership function proposed by Vasant (2003). For a value *x*, the degree of satisfaction, $\mu_{\tilde{a}_{ij}}(x)$ for fuzzy coefficient $\tilde{a}_{ij}$ is given by the membership function given in (1).

$$\mu_{\tilde{a}_{ij}}(x) = \begin{cases} 1.00 & x < a_{ij}^l \\ 0.999 & x = a_{ij}^l \\ \dfrac{B}{1 + Ce^{\alpha\left(\frac{x - a_{ij}^l}{a_{ij}^h - a_{ij}^l}\right)}} & a_{ij}^l < x < a_{ij}^h \\ 0.001 & x = a_{ij}^h \\ 0.000 & x > a_{ij}^h \end{cases} \tag{1}$$

Given a degree of satisfaction value μ, the crisp value $a_{ij}|_\mu$ for fuzzy coefficient $\tilde{a}_{ij}$ can be calculated using Eq. (2).

$$\tilde{a}_{ij}\Big|_\mu = a_{ij}^l + \left(\frac{a_{ij}^h - a_{ij}^l}{\alpha}\right)\ln\frac{1}{C}\left(\frac{B}{\mu} - 1\right) \tag{2}$$

The value α determines the shape of the membership function, whereas B and C values can be calculated from α, Eqs. (3) and (4).

$$C = -\frac{0.998}{\left(0.999 - 0.001e^{\alpha}\right)} \tag{3}$$

$$B = 0.999(1 + C) \tag{4}$$

If we wish that for a degree-of-satisfaction value μ = 0.5, the crisp value $a_{ij}\,|_{0.5}$ is in the middle of the interval

$$a_{ij|0.5} = \frac{a_{ij}^l + a_{ij}^h}{2}$$

then, α = 13.81350956.

2.3 Problem Formulation

Given a degree of satisfaction value μ, the fuzzy constrained optimization problem can be formulated (Jiménez et al., 2006; Vasant, 2004) as the non linear constrained optimization problem shown in the following:

$$\text{Maximize} \quad \sum_{i=1}^{8}\left(c_i x_i - d_i x_i^{2}\right)$$

subject to

$$\sum_{i=1}^{8}\left[a_{ij}^{l} + \left(\frac{a_{ij}^{h} - a_{ij}^{l}}{\alpha}\right)\ln\frac{1}{C}\left(\frac{B}{\mu} - 1\right)\right]x_i - b_j \le 0, \quad j = 1,\ldots,\ 17$$

$$\sum_{i=7}^{8} r_i x_i - 0.15\sum_{i=1}^{6} r_i x_i \le 0$$

$$x_1 - 0.6x_2 \le 0$$

$$x_3 - 0.6x_4 \le 0$$

$$x_5 - 0.6x_6 \le 0$$

$$0 \le x_i \le u_i, \qquad i = 1,\ldots,\ 8$$

3. A MULTI-OBJECTIVE EVOLUTIONARY APPROACH

In this section, we propose a multi-objective optimization approach to solve the problem shown above for all satisfaction degree values, which composes the fuzzy solution of the former fuzzy optimization problem. In the multi-objective optimization problem, a new input variable is considered in order to find the optimal solution for each degree-of-satisfaction value (Jimenez et al., 2004a, 2004b, 2006).

The following formulation shows the multi-objective constrained optimization problem for Chocoman Inc. In this problem, x_9 represents the degree-of-satisfaction value, which must be minimized to generate the desired Pareto front.

$$\begin{array}{ll}
\text{Maximize} & \sum_{i=1}^{8}\left(c_i x_i - d_i x_i^2\right) \\
\text{Minimize} & x_9 \\
\text{subject to} &
\end{array}$$

$$\sum_{i=1}^{8}\left[a_{ij}^{l} + \left(\frac{a_{ij}^{h} - a_{ij}^{l}}{\alpha}\right)\ln\frac{1}{C}\left(\frac{B}{\mu} - 1\right)\right]x_i - b_j \le 0, \quad j = 1, \ldots, 17$$

$$\sum_{i=7}^{8} r_i x_i - 0.15\sum_{i=1}^{6} r_i x_i \le 0$$

$$x_1 - 0.6x_2 \le 0$$

$$x_3 - 0.6x_4 \le 0$$

$$x_5 - 0.6x_6 \le 0$$

$$0 \le x_i \le u_i, \qquad i = 1, \ldots, 8$$

$$0.001 \le x_9 \le 0.999,$$

Multi-objective Pareto-based evolutionary algorithms (Coello et al., 2002; Deb, 2001; Jiménez et al., 2002) are especially appropriate to solve multi-objective nonlinear optimization problems because they can capture a set of Pareto solutions in a single run of the algorithm.

We propose an ad hoc multi-objective Pareto-based evolutionary algorithm to solve the Chocoman Inc. problem. The algorithm uses a real-coded representation, uniform and arithmetical cross, and uniform, nonuniform and minimal mutation (Jiménez et al., 2002). Diversity among individuals is maintained by using an ad hoc elitist generational replacement technique.

The algorithm has a population P of N solutions. For each solution i, f_j^i is the value for the j-th objective ($j = 1, \ldots, n$) and g_j^i is the value for the j-th constraint ($j = 1, \ldots, m$). For the Chocoman Inc. problem, $n = 2$ and $m = 21$.

Given a population P of N individuals, N children are generated by random selection, crossing, and mutation. Parents and children are ordered in N slots in the following way. A solution i belongs to slot si such that

$$s_i = \left\lfloor f_i^2 N \right\rfloor$$

The order inside slots is established with the following criteria. Position p_i of solution i is lower than position p_j of solution j in the slot if:

- i is feasible and j is unfeasible,
- i and j are unfeasible and $g^i_{\max} \le g^j_{\max}$,
- i and j are feasible and i dominates j

- i and j are feasible and nondominated and $cd_i > cd_j$

where

- $g^i_{\max} = \max j = 1,...,m \{g^i_j\}$
- and cd_i is a metric for the crowding distance of solution i:

$$cd_i = \begin{cases} \infty, & if \quad f^i_j = f^{\max}_i \quad \text{or} \quad f^i_j = f^{\min}_i \quad \text{for any } i \\ \sum_{j=1}^{n} \frac{f^{\sup_i}_j - f^{\inf_i}_j}{f^{\max}_j - f^{\min}_j}, & \text{in anothercase} \end{cases}$$

where

$$f_j^{max} = \max_{i=1,...,N} \{f_j^i\} \qquad f_j^{min} = \min_{i=1,...,N} \{f_j^i\}$$

$f_j^{\sup_j^i}$ is the value of the jth objective for the higher solution adjacent in the jth objective to i,

$f_j^{\inf_j^i}$ f is the value of the jth objective for the solution lower adjacent in the jth objective to i.

The new population is obtained by selecting the N best individual from the parent and children. The following heuristic rule is considered to establish an order. Solution i is better than solution j if:

- $p_i < p_j$
- $p_i = p_j$ and $cd_i > cd_j$

where p_i is the position of solution i in its slot.

4. EXPERIMENTS AND RESULTS

To compare performance of the algorithms in multi-objective optimization, we have followed an empirical methodology similar to that proposed in Laumanns et al. (2001) and Purshouse and Fleming (2002). It has been used as a measure v that calculates the fraction of the space that is not dominated by any of the solutions obtained by the algorithm (Laumanns et al., 2001; Zitler et al., 2003). The aim is to minimize the value of v. This measure estimates both the distance of solutions to the real Pareto front and the spread. Value v can be calculated as shown in Eq. (5) where P^t is composed by the N^t non dominated solutions of P and $f_j^{u\,max}$ and $f_j^{u\,min}$ are the utopia maximum and minimum value for the j-th objective. For the Chocoman Inc. problem, utopia minimum and maximum values are shown in Table 5.

$$\upsilon = 1 - \frac{\sum_{i=1}^{N^t} \left[\left(f_n^{u\max} - f_n^i \right) \prod_{j=1}^{n-1} \left(f_j^{\sup_j^i} - f_j^i \right) \right]}{\prod_{j=1}^{n} \left(f_j^{u\max} - f_j^{u\min} \right)} \tag{5}$$

Table 5. Utopia Minimum and Maximum Values for the Chocoman Inc. Problem.

	Utopia Min.	Utopia Max.
Objective 1	140,000	200,200
Objective 2	0.001	0.999

The parameters were set up using a previous process using a methodology similar to the one proposed in Laumanns et al. (2001). Table 6 shows the parameters obtained.

Table 6. Parameters in the Run of the Proposed Algorithm and NSGA-II for the Chocoman Inc. Problem.

Number of iterations	$T = 10000$
Population size	$N = 100$
Cross-probability	$pCross = 0.8$
Mutation probability	$pMutate = 0.5$
Uniform cross-probability	$pUniformCross = 0.7$
Uniform mutation probability	$pUniformMutate = 0.7$
Parameter c for nonuniform mutation	$c = 2.0$

Various metrics for both convergence and diversity of the populations obtained have been proposed for a more exact evaluation of the effectiveness of the evolutionary algorithms. In his book, Deb (2001) assembles a wide range of the metrics that figure in the literature. For this chapter we propose the use of two metrics to evaluate the goodness of the algorithm.

The first metric we use is the generational distance (Y) proposed by Veldhuizen and Lamont (1999), evaluates the proximity of the population to the Pareto optimal front by calculating the average distance of the population from an ideal population P^* made up of N^* solutions distributed uniformly along the Pareto front. This metric is shown in Eq. (6).

$$r = \frac{\left(\sum_{i=1}^{N'} d_i^v \right)}{N'} \qquad (6)$$

We use $v = 1$, and parameter $d\min_i$ is the Euclidean distance (in the objective space) between the solution i and the nearest solution in P^*:

$$d_i = \min_{k=1}^{N^*} \sqrt{\sum_{j=1}^{n} \left(f_j^i - f_j^{*k} \right)^2}$$

where f_j^{*k} k is the value of the j-th objective function for the k-th solution in P^*. For our problem, we use the points in Tables 7 and 8 as the ideal population P^*.

Table 7. Optimal Points for Uniformly Distributed Values In μ Obtained with Gradient for the Chocoman Inc. Problem

μ	x_1	x_2	x_3	x_4	x_5
0.001	2.397.161	3.995.268	1.989.859	3.316.432	1.411.270
0.1	2.794.339	4.657.232	2.347.411	3.912.352	1.553.713
0.2	2.875.612	4.792.687	2.420.906	4.034.844	1.583.183
0.3	2.932.170	4.886.950	2.472.111	2.472.111	1.603.752
0.4	2.980.141	4.966.902	2.515.579	4.192.632	1.621.237
0.5	3.025.503	5.042.506	2.556.712	4.261.187	1.637.803
0.6	3.072.207	5.120.344	2.599.091	4.331.819	1.654.891
0.7	3.124.697	5.207.829	2.646.756	4.411.260	1.674.136
0.8	3.191.108	5.318.513	2.707.111	4.511.852	1.698.541
0.9	3.296.295	5.493.824	2.802.814	4.671.356	1.737.323
0.999	4.143.502	6.905.837	3.540.144	5.900.239	2.000.325

Table 8. Optimal Points for Uniformly Distributed Values in μ Obtained with Gradient for the Chocoman Inc. Problem-(Continued)

μ	x_6	x_7	x_8	Profit
0.001	2.352.116	1.392.046	1.170.292	150089.2
0.1	2.589.522	1.593.681	1.673.137	165662.6
0.2	2.638.638	1.635.240	1.772.523	168585.9
0.3	2.672.920	1.664.218	1.841.035	170566.9
0.4	2.702.061	1.688.830	1.898.738	172212.8
0.5	2.729.672	1.712.133	1.952.965	173740.0
0.6	2.758.152	1.736.153	2.008.459	175282.7
0.7	2.790.226	1.763.184	2.070.434	176980.6
0.8	2.830.901	1.797.432	2.148.254	179074.1
0.9	2.895.538	1.851.788	2.270.207	182264.4
0.999	3.333.874	2.000.000	5.448.504	200116.4

The second metric we use is the spread (Δ) put forward by Deb (2001) to evaluate the diversity of the population. Equation (7) shows this measure.

$$\Delta = \frac{\sum_{j=1}^{n} d_j^e + \sum_{i=1}^{N} \left| d_i - \bar{d} \right|}{\sum_{j=1}^{n} d_j^e + N\bar{d}} \tag{7}$$

where d_i may be any metric of the distance between adjacent solutions, and $\bar{d}$ is the mean value of such measurements. In our case, d_i has been calculated using the Euclidean distance. Parameter d_j^e is the distance between the extreme solutions in P^* and P corresponding to the j-th objective function.

Table 9 shows the best, worst, medium, and variance values for the v, Y, and Δ measures obtained in 10 executions of both algorithms.

Table 9. Results of 10 Runs of the Proposed Algorithm and NSGA-II for the Chocoman

Algorithm	*vbest*	*vworst*	*vmean*	*vvariance*
Proposed algorithm	0.5366	0.583	0.5589	2.1568×10^{-5}
NSGA-II	0.5519	0.5928	0.5715	1.143×10^{-5}
	Ybest	*Yworst*	*Ymean*	*Yvariance*
Proposed algorithm	227781.6632	228187.1852	2.28031.9.239	1479.6619
NSGA-II	2.27914.8763	2.28427.1933	228228.772	2724.6756
	Δbest	*Δworst*	*Δmean*	*Δvariance*
Proposed algorithm	0.9737	0.9898	0.9837	2.8096×10^{-6}
NSGA-II	0.9735	0.9809	0.9784	6.0036×10^{-7}

Figure 1 shows the non dominated solutions obtained in the best of 10 executions of the proposed algorithm and NSGA-II for the Chocoman Inc. problem.

5. CONCLUSIONS AND FUTURE WORKS

5.1 Conclusions

Fuzzy nonlinear optimization problems are, in general, difficult to solve. In this chapter we describe a multi-objective approach to solving a fuzzy nonlinear constrained optimization problem that appears in production planning for chocolate manufacturing. A Pareto-based evolutionary algorithm is proposed to capture the solution in a single run of the algorithm. Optimality and diversity metrics have been used for the evaluation of the effectiveness of the proposed multi-objective evolutionary algorithm compared with the well-known algorithm NSGA-II. We show the values obtained using these metrics for the solutions generated by both algorithms. The results show a real ability of the proposed approach to solve problems in production planning for chocolate manufacturing.

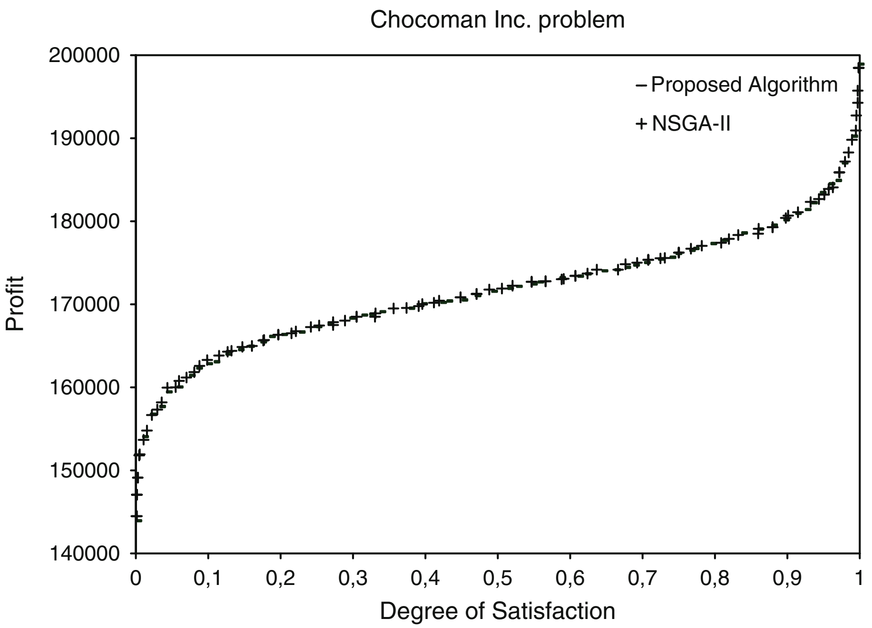

Figure 1. Nondominated solutions obtained with the proposed algorithm and NSGAII for the Chocoman Inc. problem.

5.2 Future Works

Multi-objectives with several other objective functions can be considered for future research work as well as fuzzy costs and fuzzy right-side coefficients in constraints. There is a possibility of designing a productive, computational intelligence, self-organized evolutionary fuzzy system.

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MULTI-OBJECTIVE GEOMETRIC PROGRAMMING AND ITS APPLICATION IN AN INVENTORY MODEL

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Abstract: In this chapter, first the general multi-objective geometric programming problem is defined, then Pareto optimality, the fuzzy geometric programming technique to solve a multi-objective geometric programming problem is discussed, and finally a multi-objective marketing planning inventory problem is explained and formulated. Numerical examples are given for the inventory problem in a multinational soft drink manufacturing company.

Key words: Multi-objective, geometric programming, fuzzy sets, inventory, Pareto optimality, posynomial function

1. INTRODUCTION

As society becomes more complex and as the competitive environment develops, business persons are finding that they require multiple objectives. Almost every imperative real-world problem involves more than one objective. In such cases, decision makers evaluate the best possible approximate solution alternatives according to multiple criteria.

A general multiple objective (or multiple criteria) nonlinear programming (MONLP) problem is of the following form:

Find $x = (x_{1,}\ x_{2,\ldots,} x_n)^T$(*)

which minimizes

$$F(x) = (f_1(x), f_2(x) \dots f_k(x))^T$$

subject to $g_j(x) \le b_j, (j = 1, 2, ..., m)$

and

$$x (= (x_1, x_2, ... x_k)^T) \ge 0$$

where

$f_1(x), f_2(x), ... f_k(x)$ are (≥ 2) and $g_j(x) (j = 1, 2, ..., m)$ are functions.

Here

$f_i : R^n \to R$ for $i = 1, 2, ..., k$ and $g_j : R^n \to R$ *for* $j = 1, 2, ..., m$

REMARK 1.

When $k = 1$, problem (*) reduces to a single objective NLP problem. It is noted that if the objectives of the original problem are to minimize

$$f_r(x) \quad \text{for} \quad r = k_0 + 1, k_0 + 2, ... k$$

then the objective in the mathematical formulation will be

$$\text{Minimize} \quad F(x) = (f_1 \ (x), \ f_2 \ (x) \dots, \ f_{k_0} \ (x), - \ f_{k_{0+1}} \ (x), \ - f_{k0+2} \ (x), \dots, \ - \ f_k(x))^T$$

subject to the same constraints as in (*)

If $f_r \ (x), \ (r = 1, 2, \dots, k), \ g_j(x), \ (j = 1, 2, \dots, m)$ are linear, the corresponding problem (*) is called multiple objective linear programming (MOLP) problem. When all or any one of the above functions are nonlinear, it is referred as a MONLP problem. When all of the above functions are posynomial or signomial, (*) is referred as a multi-objective geometric programming problem (MOGPP).

A MOGPP can be stated as

Find $xt = \left(x_1, x_2, \ldots, x_n\right)^T$ so as to (1)

Minimize $f_1\left(x\right) = \sum_{i=1}^{T_1^0} c_{1i}^0 \prod_{r=1}^{n} x_r^{a_{1ir}^0}$

Maximize $f_2\left(x\right) = \sum_{i=1}^{T_2^0} c_{2i}^0 \prod_{r=1}^{n} x_r^{a_{2ir}^0}$ $x > 0$

....

Minimize $f_k\left(x\right) = \sum_{i=1}^{T_k^0} c_{ki}^0 \prod_{r=1}^{n} x_r^{a_{kir}^0}$

subject to

$g_p\ \left(x\right) = \sum_{s=1}^{T_p} c_{ps} \prod_{r=1}^{n} x_r^{a_{psr}} \leq 1 \qquad p = 1, 2 \ldots\ldots., m$

where c_{ji}^0 $\left(>0\right)$, c_{ks} $\left(>0\right)$, a_{jir}^0, a_{jir} are all real numbers for j = 1, 2,.., k; i = 1, 2, .., T_j^0 ; k = 1, 2, …, m; s = 1, 2, …, T_k

Let X be a set of constraints of (*) such that

$X = \{x \in \Re^n \mid g_i(x) \leq b_j$, $j = 1, 2, \ldots, m$, and $x = (x_1, x_2, \ldots, x_n)^T$ with $x_i \geq 0$ for $i = 1, 2, \ldots, n\}$

REMARK 2.

The multi–objective optimization problem is convex if all the objective functions and the feasible region are convex.

2. PARETO OPTIMALITY

In single objective optimization problems, the main focus is on the decision variable space, whereas in the multi-objective framework, we are often more interested in the objective space (see Ehrogott, 2005). In multi-objective programming problems, multiple objectives are usually noncommensurable and cannot be combined into a single objective. In the MONLP problem, the objectives are simultaneously optimized. But due to an intrinsic conflicting nature among the objectives, it is not possible to find a single solution that would be optimal for all the objectives simultaneously. Consequently, the aim in solving MONLP is to find a compromise or satisfying solution of the decision maker. There is no natural ordering in the objective space because it is only partially ordered. For example, $(3,3)^T$ can be said to be less than $(7,7)^T$, but we cannot say any such order between $(6,2)^T$ and $(5,8)^T$.

However, some of the objective vectors can be extracted for examination. These vectors are those where none of the components can be improved without deterioration to at least one of the other components. This definition is usually called Pareto optimality, which is laid, by French-Italian economist and sociologist Vilfredo Pareto (Aliprantis et al., 2001).

DEFINITION 1.

Let x^* be the optimal solution of the following problem:

$$\text{Minimize} \quad f_r(x) \quad r = 1, 2, \ldots, k$$
$$\text{subject to} \quad x \in X$$

The point x^* is known as **ideal objective value** and rth objective function value at x^* i.e. $f_r(x^*)$ is known as ideal objective value.

DEFINITION 2.

x^* is said to be a **Complete optimal solution** to the MONLP problem (1) if there exists $x^* \in X$ such that $f_r(x^*) \le f_r(x), (r=1,2......,k)$ for all $x \in X$.

In general, the objective functions of the MONLP conflict with each other; a complete optimal solution does not always exist, and so the Pareto (or non dominated) optimality concept is introduced.

DEFINITION 3.

A decision vector $x^* \in X$ is a **Pareto optimal solution** if there does not exist another decision vector $x \in X$ such that $f_r(x) \le f_r(x^*)$ for all $r = 1, 2, \ldots, k$ and $f_{r1}(x) \ne f_{r1}(x^*)$ for at least one $r_1 = 1, 2, \ldots, k$.

An objective vector F^* is Pareto optimal if there does not exist another objective vector $F(x)$ such that $f_r \le f_r^*$ for all $r = 1, 2, ...k$ and $f_{r1} < f_{r1}^*$ for at least one index r_1. Therefore, F^* is Pareto-optimal if the decision vector corresponding to it is Pareto optimal.

REMARK 3.

In general, a Pareto optimal solution consists of an infinite number of solutions. A Pareto optimal solution is sometimes called a noninferior solution since it is not inferior to other feasible solutions.

DEFINITION 4.

A decision vector $x^* \in X$ is a **weakly Pareto optimal solution** if there does not exist another decision vector $x \in X$ such that $f_r(x) < f_r(x^*)$ for all $r = 1, 2, \ldots, k$.

DEFINITION 5.

$x^* \in X$ is said to be a **locally Pareto optimal solution** to the MONLP if and only if there exists an $r < 0$ such that x^* is Pareto optimal in $X \cap N(x^*, r)$; i.e. there does not exist another $x^* \in X \cap N(x^*, r)$ such that $f_i(x) \leq f_i(x^*)$.

Now, we introduce some non linear programming techniques, which have been used in this thesis to achieve at least local Pareto optimal solutions.

2.1 Method of Global Criterion

In this method, the distance between some reference point and the feasible objective region is minimized. The decision maker has to select the reference point and the metric for measuring the distances. In this way, the multiple objective functions are transferred into a single objective function. We suppose that the weighting coefficients ω_r are real numbers such that $\omega_r \geq 0, \quad \forall_r \; = \; 1,2,\ldots,k$ and

$$\sum_{r=1}^{k} \omega_r = 1$$

The weighted L_p-problem for minimizing distances is stated as

$$\text{Minimize} \qquad L_p(f(x)) = \left(\sum_{r=1}^{k} \omega_r \, | f_r(x) - f_r(x^*) |^p \right)^{\frac{1}{p}} \tag{2}$$

subject to $\quad x \in X$, for $1 \leq p < \infty$

2.2 Hybrid Method

Following Chankong and Haimes (1983), the hybrid problem combining L_p and the ε-constraint method is as follows:

$$\text{Minimize} \quad L_p(f(x)) = \left(\sum_{r=1}^{k} \omega_r \mid f_r(x) - f_r(x^*) \mid^p \right)^{1/p} \tag{3}$$

subject to $f_r \leq \omega_r$

$x \in X$, for $1 \leq p < \infty$

where $\omega_r \geq 0, \quad \forall r = 1, 2, \ldots, k \qquad \sum_{r=1}^{k} \omega_r = 1$

and $\varepsilon_r (\leq f_r(x^*))$ is a real number for all $r = 1, 2, \ldots, k$.

$$\text{For } p = 1, \quad L_1(f(x)) = \sum_{r=1}^{k} \omega_r |f_r(x) - f_r(x^*)| \tag{4}$$

The objective function $L_1(f(x))$ is the sum of the weighted deviations, which is to be minimized and is known as weighted sum method.

$$\text{For } p = 2, \; L_2(f(x)) = \left(\sum_{r=1}^{k} \omega_r \mid f_r(x) - f_r(x^*) \mid^2 \right)^{1/2} \tag{5}$$

When p becomes larger, the minimization of the deviation becomes more and more important.

Finally, when $p \rightarrow \infty$, the only thing that matters is the weighted relative deviation of one objective function; i.e.,

$$L_\infty(f(x)) = \underset{r=1,2,\ldots,k}{\text{Max}} \left(\sum_{r=1}^{k} \omega_r \mid f_r(x) - f_r(x^*) \mid \right) \tag{6}$$

This multi-objective method is called the “min–max” method or the Tchebycheff method. Problem (6) is nondifferentiable like its unweighted counterpart. Correspondingly, it can be solved in a differentiable form as long as the objective and the constraint functions are differentiable and $f_r(x^*)$ is known globally. In this case, instead of problem (6), the problem becomes

$$\text{Minimize } \lambda$$

$$\text{subject to } \omega_r \left(f_r(x) - f_r(x^*)\right) \leq \lambda \text{ for all } r = 1, 2, \ldots, k \tag{7}$$

$$x \in X, \quad \lambda \in \Re$$

THEOREM 6.

The solution of weighted L_p–problem (when $1 \leq p < \infty$) is a Pareto optimal solution if all the weighting coefficients are positive.

THEOREM 7.

The solution of a weighted Tchebycheff problem (L_∞) is weakly Pareto optimal if all the weighting coefficients are positive.

THEOREM 8.

The weighted Tchebycheff problem has at least one Pareto optimal solution.

THEOREM 9.

Let a decision vector $x^* \in X$ be given, Solve the problem

$$\text{Minimize } \sum_{r=1}^{k} f_r(x) \tag{8}$$

$$\text{subject to } f_r(x) \leq f_r(x^*) \text{ for all } r = 1, 2, \ldots, k \text{ and } x \geq 0.$$

Let Ø(x^*) be the optimal objective value. The decision vector $x^* \in X$ is Pareto optimal if and only if it is a solution of Eq. (8) so that

$$\phi(x^*) = \sum_{r=1}^{k} f_r(x^*)$$

Proof.

The proof of Theorems 9−12 are followed by Miettinen (1999).

When $f_r(x)$ and (r = 1, 2,…, k) and $g_j(x)$ (j = 1, 2, …, k) are polynomial and signomial functions, Eqs. (4), (5), and (7) may be reduced to a single objective geometric programming problem.

3. FUZZY GEOMETRIC PROGRAMMING TECHNIQUE TO SOLVE A MULTI-OBJECTIVE GEOMETRIC PROGRAMMING PROBLEM

Multi-objective geometric programming (MOGP) is a special type of a class of MONLP problems. Biswal (1992) and Verma (1990) developed a fuzzy geometric programming technique to solve a MOGP problem. Here, we have discussed a fuzzy geometric programming technique based on max−min and max−convex combination operators to solve a MOGP.

When $f_r(x)$ (r = 1, 2,…, k) and g_i (x) (j = 1, 2,…, m) are polynomial or signomial functions, Eq. (1) may be taken as a MOGP.

To solve the MOGP problem (1), we use the Zimmerman's (1978) technique. The procedure consists of the following steps.

Step 1. Solve the MOGP as a single objective GP problem using only one objective at a time and ignoring the others. These solutions are known as ideal solutions. Repeat the process k times for k different objective functions. Let x^1, x^2, x^3,…, x^k be the ideal solutions for the respective objective functions, where

$$x^r = (x_1^r, x_2^r, \ldots, x_n^r)$$

Step 2. From the ideal solutions of Step 1, determine the corresponding values for every objective at each solution derived. With the values of all objectives at each ideal solution, the pay-off matrix of size ($k \times k$) can be formulated as follows :

$$f_1(x) \quad f_2(x) \quad \ldots \quad f_k(x)$$

$$\begin{array}{c} x^1 \\ x^2 \\ \cdots\cdots \\ x^k \end{array} \begin{pmatrix} f_1^*(x^1) & f_2(x^1) & \cdots\cdots & f_k(x^1) \\ f_1(x^2) & f_2^*(x^2) & \cdots\cdots & f_k(x^2) \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \\ f_1(x^k) & f_2(x^k) & \cdots\cdots & f_k^*(x^k) \end{pmatrix}$$

Step 3. From the Step 2, find the desired goal L_r and worst tolerable value U_r of $f_r(x)$, $r = 1, 2, \ldots, k$ as follows:

$$L_r \le f_r \le U_r, \ (r = 1, 2, \ldots, k)$$

where

$$U_r = \text{Max } (f_r(x^1), f_r(x^2), \ldots, f_r(x^{(r-1)}), f_r^*(x^r), f_r(x^{(r+1)}), \ldots, f_r(x^k))$$

$$L_r = \text{Min } (f_r(x^1), f_r(x^2), \ldots, f_r(x^{(r-1)}), f_r^*(x^r), f_r(x^{(r+1)}), \ldots, f_r(x^k))$$

Step 4. Define a fuzzy linear or non-linear membership function $\mu_r(f_r(x))$ for the r-th objective function $f_r(x), (r = 1, 2, 3, \ldots k)$

$$\mu_r(f_r(x)) = \begin{cases} 1 & \text{if} \quad f_r(x) \le L_r \\ u_r(f_r(x)) & \text{if} \quad L_r \le f_r(x) \le U_r \\ 0 & \text{if} \quad f_r(x) \ge U_r \end{cases}$$

Here $u_r(f_r(x))$ is a strictly monotonic decreasing function with respect to $f_r(x)$.

Step 5. At this stage, either a max−min operator or a max−convex combination operator can be used to formulate the corresponding single objective optimization problem.

3.1 Through a Max−Min Operator

According to Zimmermann (1978), the problem (1) can be solved as:

$$\mu_D(x^*) = \text{Max}(\text{Min}\ (\mu_1\ (f_1(x)),\ \mu_2(f_2(x)),\ ...,\ \mu_k\ (f_k(x)))) \quad (9)$$

subject to

$$g_j(x) \le b_j, \quad j = 1,2,\ldots,m \qquad x > 0$$

which is equivalent to the following problem as

$$\text{Maximize} \ \ \alpha \qquad (10)$$

subject to

$$\alpha \le \mu_r(f_r(x)), \quad \text{for} \quad r = 1,2,3,\ldots,k$$

$$g_j(x) \le b_j, \quad \text{for} \quad j = 1,2,\ldots,m \qquad x > 0$$

The parameter α is called an aspiration level and represents the compromise among the objective functions. After reducing the problem (10) into a standard form of a PGP problem, it can be solved through a GP technique.

3.2 Through a Max–Convex Combination Operator

Using the membership functions $\mu_r(f_r(x))$ to formulate a crisp non-linear programming model (following Tiwari et al., 1987) by adding the weighted membership functions together as:

$$\mu_D(x^*) = \text{Maximize}\left(\sum_{r=1}^{m} \lambda_r \mu_r(f_r(x))\right) \qquad (11)$$

subject to

$$g_i(x) \le b_j, \quad j=1,\ 2,\ldots,\ m \qquad x > 0.$$

For equivalent weights, $\lambda_1 = \lambda_2 = \ldots = \lambda_m = 1$ are considered.

EXAMPLE 10.

Solve MOGP

$$\text{Minimize} \quad [Z_1(x), Z_2(x)] \tag{12}$$

subject to

$$Y(x) \le 1, \quad x > 0$$

where

$Z_1(x) = 25x_1^2 + 30x_1^{-2}x_2$, $Z_2(x) = 15x_1 + 20x_1^{-1}x_2^2$, $Y(x) = x_1^{-1}x_2^{-1}$ and $x = (x_1, x_2)^T$.

In order to solve the problem (12), we shall have to solve the sub-problems

$$\text{Minimize} \quad Z_1(x) \tag{13}$$

subject to

$Y(x) \le 1$ (Sub-PGP – 1), $x > 0$

It is a GP with DD = 3 – (2 + 1) = 0 and

$$\text{Minimize} \quad Z_2(x) \tag{14}$$

subject to

$Y(x) \le 1$ (Sub-PGP–2), $x > 0$

It is a GP with DD = 3 – (2+1) = 0

Solving the sub-problems (12) and (13) by GP technique, we have

For (Sub PGP – 1) $x^{1*} =$ (1.124746, 0.8890896) and $Z_1^*(x^*) = 52.70158$.

For (Sub PGP – 2) $x^{2*} =$ (1.414219, 0.7071040) and $Z_2^*(x^*) = 28.28427$.

The pay-off matrix is given below:

$$\begin{matrix} \\ x^1 \\ x^2 \end{matrix}\begin{pmatrix} Z_1(x) & Z_2(x) \\ 52.71058 & 30.92735 \\ 60.60687 & 28.28427 \end{pmatrix}$$

From pay-off matrix the lower and upper bounds of $Z_1(x)$ be 52.71058 and 60.60687 and that of $Z_2(x)$ be 30.92735 and 28.28427.

[$\because$ 52.71058 $\le Z_1(x) \le$ 60.60687 and 28.28427 $\le Z_2(x) \le$ 30.92735]

Suppose $\mu_{Z1}(x)$ and $\mu_{Z2}(x)$ are the linear membership functions of the objective functions $Z_1(x)$ and $Z_2(x)$ respectively and they are defined as:

$$\mu_{Z1}(x) = \begin{cases} 1, & \text{if } Z_1(x) \le 52.71058 \\ \dfrac{60.60687 - Z_1(x)}{7.89629} & \text{if } 52.71058 \le Z_1(x) \le 60.60687 \\ 0 & \text{if } Z_1(x) \ge 60.60687 \end{cases}$$

$$\mu_{Z2}(x) = \begin{cases} 1, & \text{if } Z_2(x) \le 28.28427 \\ \dfrac{30.92735 - Z_2(x)}{2.64308} & \text{if } 28.28427 \le Z_2(x) \le 30.92735 \\ 0 & \text{if } Z_2(x) \ge 30.92735 \end{cases}$$

Equation (14) can be reduced to a single objective GPP by max-min operator or max-addition operator.

3.3 Through a Max−Min Operator

Using max-min operator MOGP (14) can be reduced to a following single objective problem

Maximize α (15)

subject to

$$\mu_{Z_1}(x) = \frac{60.60687 - Z_1(x)}{7.89629} \geq \alpha, \ \mu_{Z_2}(x) = \frac{30.92735 - Z_2(x)}{2.64308} \geq \alpha,$$

$$x_1^{-1} x_2^{-1} \leq 1, \qquad \alpha, x_1, x_2 > 0, \text{ and } \alpha < 1$$

In the standard form of GP the problem (15) can be written as

$$\text{Minimize } \alpha^{-1} \tag{16}$$

subject to

$$0.41249\, x_1^2 + 0.49499\, x_1^{-2} x_2 + 0.13029\, \alpha \leq 1$$
$$0.48501\, x_1 + 0.64668\, x_1^{-2} x_2^2 + 0.08546\, \alpha \leq 1$$
$$x_1^{-1} x_2^{-1} \leq 1, \qquad \alpha, x_1, x_2, > 0, \qquad \text{and} \qquad \alpha < 1$$

The problem (16) has DD = 8–(3+1)=4. The corresponding dual problem (DP) is

Maximize $d(w)=$

$$\left(\frac{1}{w_1}\right)^{w_1} \left(\frac{0.41249\gamma_A}{w_2}\right)^{w_2} \left(\frac{0.49499\gamma_A}{w_3}\right)^{w_3} \left(\frac{0.13029\gamma_A}{w_4}\right)^{w_4} \tag{17}$$

$$\left(\frac{0.48501\gamma_B}{w_5}\right)^{w_5} \left(\frac{0.64668\gamma_B}{w_6}\right)^{w_6} \left(\frac{0.08546\gamma_B}{w_7}\right)^{w_7} (1)^{w_8}$$

subject to the following normal and orthogonal conditions are as follows:

$$w_1 = 1$$

$$-w_1 + w_4 + w_\gamma = 1, \ 2w_2 - 2w_3 + w_5 - 2w_6 - w_8 = 0$$

$$w_3 + 2w_6 - w_8 = 0$$

where

$$\gamma_A = w_2 + w_3 + w_4, \gamma_B = w_5 + w_6 + w_7$$

$$0 < w_1, w_2, w_3, w_4, w_5 \leq 1$$

Solving the DP (17) subject to the normal and orthogonal conditions, we get the optimal values of dual variables $w_1^* = 1$, $w_2^* = 0.54598$, $w_3^* = 0.30614$, $w_4^* = 0.12419$, $w_5^* = 0.55135$, $w_6^* = 0.34351$, $w_7^* = 0.08146$, and $w_8^* = 0.976311$. The optimal dual objective value is $d(w^*) = 1.02426$, and hence, the optimal values of the decision variables are $x_1^* = 1.16436$ and $x_1^* = 0.85884$. Then $Z_1^*(x^*) = 52.89796$ and $Z_2^*(x^*) = 30.13521$.

3.4 Through a Max–Convex Combination Operator

Using a convex-combination operator, the multi-objective problem (12) can be transformed into a following single objective problem:

$$\text{Maximize } V(\mu_{Z_1}(x),\ \mu_{Z_2}(x)) = \left(\lambda_1 \mu_{Z_1}(x) + \lambda_2 \mu_{Z_2}(x)\right) = 19.37661 - g(x) \quad (18)$$

subject to

$$x_1^{-1} x_2^{-1} \leq 1 \qquad x_1,\ x_2 > 0$$

where

$$g(x) = 3.166604\, x_1^2 + 3.79925\, x_1^{-2} x_2 + 5.67520\, x_1 + 7.56693\, x_1^{-1} x_2^2$$

$$\left[\text{Here } \lambda_1 = \lambda_2 = 1.\right]$$

For maximizing the problem (18), it is sufficient to solve the following problem:

Minimize $g(x) =$

$$3.16604\, x_1^2 + 3.79925\, x_1^{-2} x_2 + 5.67520\, x_1 + 7.56693\, x_1^{-1} x_2^2 \quad (19)$$

subject to

$$x_1^{-1}x_2^{-1} \le 1,\ x_1,\ x_2 > 0$$

The problem has DD = 5 – (2+1) = 2. The corresponding dual problem is

Maximize $d(w) =$

$$\left(\frac{3.16604}{w_1}\right)^{w_1}\left(\frac{3.79925}{w_2}\right)^{w2}\left(\frac{5.67520}{w_3}\right)^{w3}\left(\frac{7.56693}{w_4}\right)^{w_4}(1)^{w5} \qquad (20)$$

subject to the normal and orthogonal conditions

$w_1 + w_2 + w_3 + w_4 = 1,\qquad 2w_1 - 2w_2 + w_3 - w_4 - w_5 = 0$

$w_2 + 2w_4 - w_5 = 0,\qquad 0 < w_1, w_2, w_3, w_4, w_5 \le 1$

Solving the problem (20), we ultimately get $w_1^* =$ 0.28032, $w_2^* =$ 0.10699, $w_3^* =$ 0.39960, $w_4^* =$ 0.21309, and $w_5^* =$ 0.05599. The value of $d(w^*) =$ 17.85902. Therefore the value of $g^* =$ 17.85902 and the value of $x_1^* =$ 1.25747 and $x_2^* =$ 0.795245, and the value of objectives are $Z_1^*(x^*) =$ 54.61878 and $Z_2^*(x^*) =$ 28.92060.

4. MULTI-OBJECTIVE MARKETING PLANNING INVENTORY PROBLEM

In most inventory problems, the unit price of an item in considered as independent in nature. Actually, it relates to the demand of that item. When the demand of an item is high, it is produced in large numbers. Fixed costs of production are spread over a large number of items. Hence the unit cost of the item decreases; i.e., the unit price of an item inversely relates to the demand of that item. Cheng (1989) formulated the EOQ problem with this idea and solved it through the GP method.

Similarly the marketing cost, which includes the advertisement and promotion cost, directly affects the demand of an item. The manufacturing companies increase the advertisement cost and give some advantages (like promotion, incentives) to their sales representatives according to their performances. Lee and Kim (1993) studied the marketing planning problem considering such how to solve the problem by the GP method.

The following basic assumptions are used in the proposed model:

ASSUMPTIONS.

1. Production is instantaneous.
2. Demand is uniform.
3. The demand of a function is directly proportional to the marketing expenditure; i.e., $D_i = d_{2i} M_i^{\eta_i}, d_{2i} > 0, \eta_i > 0,$
4. The unit cost is inversely proportional to demand; i.e., $c_{0i} = c_{0i}' D_i^{-T_i}, c_{0i}' > 0$.

Let for the amount of stock is R_i at time $t = 0$. In the interval $(0, T_i = t_{1i} + t_{2i})$, the inventory level gradually decreases to meet demands. By this process, the inventory level reaches zero level at time t_{1i} and then shortages are allowed to occur in the interval (t_{1i}, T_i). The cycle then repeats itself. The differential equation for the instantaneous inventory $q_i(t)$ at time t in $(0, T_i)$ is given by

$$\frac{dq_i(t)}{dt} = -D_i \quad \text{for } 0 \le t \le T_i \tag{21}$$

with the initial conditions

$$q_i(0) = R_i,\ q_i(T_i) = -S_i \text{ and } q_i(t_{1i}) = 0.$$

For each period, a fixed amount of shortage is allowed and there is a penalty cost c_{2i} per items of unsatisfied demand per unit time. From Eq. (21)

$$q_i(t) = R_i - D_i t \quad \text{for } 0 \le t \le t_{1i}$$

$$= D_i(t_{1i} - t) \quad \text{for } t_{1i} \le t \le T_i$$

So,

$$D_i t_{1i} = R_i,\ S_i = D_i t_{2i},\ Q_i = D_i T_i$$

$$\text{Holding cost} = h_i c_{0i} \int_0^{t_{1i}} q_i(t)dt = \frac{h_i c_{0i}(Q_i - S_i)^2}{2Q_i} T_i$$

$$\text{Shortage cost} = c_{2i} \int_{t_{1i}}^{T_i} (-q_i(t))dt = \frac{c_{2i} S_i^2}{2Q_i} T_i$$

Production cost = $c_{0i}Q_i = c_{0i} d_{2i}^{-r_i} M_i^{-r_i\eta_i} Q_i$

Advertisement cost = $M_i Q_i$

The total inventory cost = setup cost + holding cost + shortage cost

$$= h_i c_{0i} \frac{(Q_i - S_i)^2}{2Q_i} T_i + c_{3i} + c_{2i} \frac{S_i^2}{2Q_i} T_i$$

The total average inventory cost,

$TC_1(M,Q,S) =$

$$\sum_{i=1}^{n} \left[\frac{h_i (Q_i - S_i)^2}{2Q_i} c_{0i} d_{2i}^{-T_i} M^{-T_i\eta_i} + c_{3i} \frac{d_{2i} M_i^{\eta_i}}{Q_i} + c_{2i} \frac{S_i^2}{2Q_i} \right] \tag{22}$$

$$= \sum_{i=1}^{n} \left[\begin{array}{l} \frac{1}{2} h_i c_{0i} d_{2i}^{-T_i} M_i^{-T_i\eta_i} Q_i - h_i c_{0i} d_{2i}^{-T_i} M_i^{-T_i\eta_i} S_i \\ + \frac{h_i c_{0i} d_{2i}^{-T_i} M_i^{-T_i\eta_i} S_i^2}{2Q_i} + c_{3i} \frac{d_{2i} M_i^{\eta_i}}{Q_i} + c_{2i} \frac{S_i^2}{2Q_i} \end{array} \right]$$

And total additional cost = marketing cost + production cost = $\sum_{i=1}^{n} [M_i Q_i + c_i Q_i]$.

So, the total average additional cost

$$TC_2(M) = \left[\sum_{i=1}^{n} d_{2i} M_i^{\eta_i + 1} + d_{2i}^{1-T_i} c_{0i} M_i^{\eta_i (1-r_i)} \right] \tag{23}$$

<u>Special Case.</u>

When shortages are not allowed i.e., when $c_{2i} \rightarrow \infty$, then

$$TC_1'(M,Q) = \sum_{i=1}^{n} \left[\frac{1}{2} h_i c_{0i} d_{2i}^{-T_i} M_i^{-T_i\eta_i} Q_i + \frac{c_{3i} d_{2i} M_i^{\eta_i}}{Q_i[\ ]} \right]. \tag{24}$$

$$TC_1'(M) = \sum_{i=1}^{n} \left[d_{2i} M_i^{\eta_i + 1} + d_{2i}^{1-r_i} c_{0i} M_i^{\eta_i (1-r_i)} \right] \tag{25}$$

4.1 Problem Formulation

The manufacturing organization produces some items and stocks these items in a warehouse. The manufacturing companies or organizations use a huge advertisement for their products in order to increase the level of demand. Still they have some limitations regarding total space capacity, total allowable shortage cost, etc. In this phenomenon, the organization is interested in minimizing the inventory-related cost (including setup cost, shortage cost) and additional cost (including marketing cost and production cost) simultaneously.

The problem is to minimize total average inventory costs and also to minimize total average of additional cost under the limitations of space capacity, total allowable shortage cost. Hence the problem is

$$\text{Minimize}\left[TC_1(M,Q,S),TC_2(M)\right] \tag{26}$$

subject to

$$\sum_{i=1}^{n} W_i(Q_i - S_i) \le W, \qquad \sum_{i=1}^{n} \frac{c_{2i}}{2}\frac{S_i^2}{Q_i} \le S$$

M_i, Q_i, S_i, > 0 for i = 1,2,…,n.

4.2 Solution Procedure of Multi-objective Inventory Model (MOIM)

The MOIM may be solved by several techniques. Some of those are the fuzzy geometric programming technique and global criterion method. Here global criterion is used to find the compromise solution of model (26). In this method, the objective functions are combined to a single objective function.

4.2.1 Global Criterion Method

Let $w_r \ge 0,\ r = 1,2$ be the normalized weights (i.e., $w_1 + w_2 = 1$) corresponding to the objective functions $TC_1(M,Q,S)$ and $TC_2(M)$. TC_{01} and TC_{02} are the ideal objective values of $TC_1(M,Q,S)$ and $TC_2(M)$, respectively. Deductions are shown in Appendix A. TC_{01} and TC_{02} are obtained objective functions for $TC_1(M,Q,S)$ and $TC_2(M)$, respectively, without constraints by GP methods. The weighted L_p-problem according to Miettinen (1999) is

$$\text{Minimize } U_p(M,Q,S) = \left(\sum_{r=1}^{2} w_r \,|[TC_r(.) - TC_{0r}\,|^p\right)^{1/p} \quad 1 \le p << \infty$$

$$= \left(w_1\,|\,TC_1(M,Q,S) - TC_{01}\,|^p + w_2\,|\,TC_2(M) - TC_{02}\,|^p\right)^{1/p} \tag{27}$$

subject to same constraints as in Eq. (26)

CASE 1.

The weighted sum problem (i.e., for p = 1 in (27) is given as

$$\text{Minimize } U_1(M,Q,S) = w_1(TC_1(M,Q,S) - TC_{01}) + w_2(TC_2(M) - TC_{02}))$$

subject to same constraints as in Eq. (26).

Since w_1, w_2, TC_{01} and TC_{02} are independent of the decision variable, so it is enough to solve the following problem:

$$\text{Minimize } V_1(M,Q,S) = w_1 TC_1(M,Q,S) + w_2 TC_2(M) \tag{28}$$

subject to same constraints as in (26)

where $U_1(M,Q,S) = V_1(M,Q,S) - (w_1 TC_{01} + w_2 TC_{02})$.

The problem is a signomial GP problem with DD = 6n – 1 and can be solved by the GP method.

CASE 2.

The least–squares problem (i.e., for $p = 2$) is given as

$$\text{Minimize} \quad U_2(M,Q,S) = \begin{bmatrix} w_1(TC_1(M,Q,S) - TC_{01})^2 + \\ w_2(TC_2(M) - TC_{02})^2 \end{bmatrix}^{1/2}$$

subject to same constraints as in Eq. (26).

To obtain the standard form of a GP problem of the above−weighted quadratic problem, we introduce two new variables y_1 and y_2, which are the upper bounds of $TC_1(.) - TC_{01}$ and $TC_2(.) - TC_{02}$ respectively (i.e., $TC_1(M,Q,S) - TC_{01} \le y_1$ and $TC_2(M) - TC_{02} \le y_2$). We may then rewrite the problem as:

$$\text{Minimize } V_2(M,Q,S,Y) = w_1 y_1^2 + w_2 y_2^2 \tag{29}$$

subject to

$$\frac{TC_1(M,Q,S)}{TC_{01}} - \frac{y_1}{TC_{01}} \le 1, \qquad \frac{TC_2(M)}{TC_{02}} - \frac{y_2}{TC_{02}} \le 1$$

$$\frac{1}{w} \sum_{i=1}^{n} W_i Q_i - S_i) \le 1, \qquad \frac{1}{s} \sum_{i=1}^{n} \frac{c_{2i} S_i^2}{2Q_i} \le 1$$

$$M_i, Q_i, S_i, y_1, y_2 > 0 \text{ for } \quad i = 1, 2, .., j$$

where

$$U_2(M,Q,S) = (V_2(M,Q,S))^{1/2}$$

The problem (29) is also a signomial GP problem with DD = 6*n* + 1 and it can be solved by the GP method.

CASE 3.

The Tchebycheff problem (i.e., for $p \to \infty$) is given as

$$\underset{M,Q,\ S}{\text{Minimize}} \left(\underset{r\ =\ 1,2}{\text{Maximize}} \left[w_r \ | \ TC_r(.) \ - \ TC_{0r} \ | \right] \right)$$

subject to same constraints as in Eq. (26) .

We introduce a new variable v, which is maximum between

$w_1(TC_1(M,Q,S) - TC_{01})$ and $w_2(TC_2(M) - TC_{02})$

(i.e., $w_1(TC_1(M,Q,S) - TC_{01}) \le v$ and

$w_2(TC_2(M) - TC_{02}) \le v$).

The problem is then reorganized as

Minimize v (30)

subject to

$$\frac{TC_1(M,Q,S)}{TC_{01}} - \frac{v}{w_1 TC_{01}} \le 1, \qquad \frac{TC_2(M)}{TC_{02}} - \frac{v}{w_2 TC_{02}} \le 1$$

$$\frac{1}{w}\sum_{i=1}^{n} W_i(Q_i - S_i) \le 1, \qquad \frac{1}{s}\sum_{i=1}^{n} \frac{c_{2i}S_i^2}{2Q_i} \le 1$$

$$y_1, y_2, M_i, Q_i, S_i > 0, \qquad i = 1, 2, \ldots, n$$

The problem is again a signomial GP problem with DD = $6n + 1$ and it can be solved by the GP technique.

4.3 Numerical Illustration

A multinational soft drink manufacturing company produces two types of brands. The brands are produced in lots. The pertinent data for the items are given in Table 1.

Table 1. Input Data for Model 17

Brands names	A	B
Inventory holding cost rate (h_i)	25%	32%
Shortage cost (c_{2i}) (\$)	10	14
Set up cost (c_{2i}) (\$)	130	150
Annual demand (D_i)	$10M_1^{1.5}$	$12M_2^{1.2}$
Production cost (c_{0i}) (\$)	$5.05M_1^{-1.8}$	$4.34M_2^{-1.464}$
Storage area (w_i) (m^2)	3	2.5

Total available storage area and total allowable shortage cost are $w = 225\ m^2$ and $S = \$\ 0.085$

The ideal value (as computed in Appendix A) of $TC_1(M,Q,S)$ is TC_{01} = \$123.1274 and that of $TC_2(M)$ is TC_{02} = \$122.2257. The company decides to know the optimal values of the inventory related cost ($TC_1(M,Q,S)$), additional cost ($TC_2(M)$), marketing cost M_1,M_2, lot sizes Q_1,Q_2 , shortage amount S_1, S_2.

Optimal solutions of problem (26) are given in Table 2, Table 3, and Table 4 for different preference values of the objective functions.

Table 2. Equal Preference Values of the Objective Functions i.e., for $(w_1, w_2) = (0.5, 0.5)$

p	i	M_i^*	Q_i^*	S_i^*	$TC_1^*(M^*,Q^*,S^*)$	$TC_2^*(M^*)$
1	1	0.9300382	39.51303	0.5809763	128.0386	123.0328
	2	0.9025289	43.79591	0.5143728		
2	1	0.9196067	37.23608	0.5836205	127.6902	123.6351
	2	0.9720086	46.52790	0.5108576		
∞	1	0.8458588	32.01543	0.5797730	127.5276	125.2465
	2	1.0992430	52.77223	0.4950188		

The above table gives different optimal solutions when the decision maker supplies equal preferences to the inventory-related cost function $TC_1(M,Q,S)$ and additional cost function $TC_2(M)$. $TC_1^*(M^*,Q^*,S^*)$ is minimum when $p \to \infty$, whereas $TC_2^*(M^*)$ is minimum when $p = 1$.

Table 3. More Preference Values to the Inventory Related Cost Functions, i.e., for $(w_1, w_2) = (0.6, 0.4)$

p	i	M^*	Q^*	S^*	$TC_1^*(M^*,Q^*,S^*)$ (\$)	$TC_2^*(M^*)$(\$)
1	1	0.9306990	38.83385	0.5819986	127.8774	123.2292
	2	0.9257421	44.61137	0.5135849		
2	1	0.9123236	36.57991	0.5838419	127.6460	123.7997
	2	0.9891751	47.31423	0.5095112		
∞	1	0.8458587	32.01543	0.5797731	127.5276	125.2465
	2	1.099243	52.77223	0.4950187		

Table 3 shows different optimal solutions when the decision maker supplies more preference to the inventory-related cost function $TC_1(M,Q,S)$ than the additional cost function $TC_2(M)$. Here $TC_1^*(M^*,Q^*,S^*)$ is minimum when $(p \to \infty)$, whereas $TC_2^*(M^*)$ is minimum when $p = 1$.

Table 4. More preference values to the Additional Cost Functions i.e for $(w_1, w_2) = (0.3, 0.7)$)

p	i	M^*	Q^*	S^*	$TC_1^*(M^*, Q^*, S^*)$ (\$)	$TC_2^*(M^*)$(\$)
1	1	0.8966431	38.19715	0.5651637	128.8857	122.5162
	2	0.8438916	40.85587	0.5020723		
2	1	0.9293127	38.46130	0.5824761	127.8178	123.3277
	2	0.9373176	45.05847	0.5130581		
∞	1	0.8961108	35.32355	0.5837589	127.5862	124.1366
	2	1.020650	48.81861	0.5063539		

The Table 4 shows different optimal solutions when the decision maker supplies more preference to the additional cost function $TC_2(M)$ than to the inventory-related cost function $TC_1(M,Q,S)$. Here TC_1^* is minimum when $(p \to \infty)$, whereas $TC_2^*(M^*)$ is minimum when $p = 1$.

5. CONCLUSION

Here we have discussed multi-objective geometric programming based on the global criterion method and then fuzzy geometric programming technique. We have also formulated the multi-objective inventory optimization model of the economic production and the marketing planning problem. The different objective functions are combined into a single objective function by the global criterion method. The GP technique is used to derive the optimal solutions for different preferences on objective functions. In Tables 2–4 we have shown the optimal solution of our problem for different preference values of the objective functions. This multiobjective inventory model may also be solved by the fuzzy geometric programming technique.

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APPENDIX A

Working rule for finding the ideal objective values TC_{01} and TC_{02}.

Minimize $TC_{1i}(M_i, Q_i, S_i) = \frac{1}{2} h_i c_{0i}^{'} d_{2i}^{-\tau i} M_i^{-\tau i \eta i} Q_i -$

$$h_i c_{0i}^{'} d_{2i}^{-\tau} M_i^{-\tau i \eta i} S_i + \frac{h_i c_{0i}^{'} d_{2i}^{-\tau i} M_i^{-\tau i \eta i} S_I^2}{2Q_i}$$

$$+ c_{3i} \frac{d_{2i} M_i^{\eta i}}{Q_i} + c_{2i} \frac{S_i^2}{2Q_i}$$

subject to $M_i, Q_i, S_i > 0$.

The above problem is a primal GP problem with DD = 1 0. The corresponding dual programming problem is

Maximize $dw_i^{'} = \left(\frac{h_i c_{0i}^{'} d_{2i}^{-Ti}}{2w1i}\right)^{w1i} \left(\frac{h_i c_{0i}^{'} d_{2i}^{-Ti}}{w2i}\right)^{w2i}$

$$\left(\frac{h_i c_{0i}^{'} d_{2i}^{-Ti}}{2w3i}\right)^{w3i} \left(\frac{c_{3i} d_{2i}}{w4i}\right)^{w4i} \left(\frac{c_{2i}}{2w_{5i}}\right)^{w5i}$$

subject to normally and orthogonal conditions

$$w_{1i} - w_{2i} + w_{3i} + w_{4i} + w_{5i} = 1$$

$$-\tau_i \eta_i w_{1i} + \tau_i \eta_i w_{2i} - \tau_i \eta_i w_{3i} + \eta_i w_{4i} = 0$$

$$w_{1i} - w_{3i} - w_{4i} - w_{5i} = 0$$

$$-w_{2i} + 2w_{3i} + 2w_{5i} = 0$$

$$w_{1i}, w_{2i}, w_{3i}, w_{4i}, w_{5i} > 0$$

Solving the dual weights in terms of w_{3i}, we get

$$w_{1i} = w_{3i} + \frac{2\tau_i - 1}{2\tau_i}, w_{2i} = 2w_{3i} + \frac{\tau_i - 1}{\tau_i}, w_{4i} = \frac{1}{2}, w_{5i} = \frac{\tau_i - 1}{2\tau_i}$$

Since the dual weights are always positive, $\tau_i > 1$. Substituting the above dual weights into the dual function and then differentiating them with respect to w_{3i}, we get

$$w_{3i}^* = \frac{(\tau_i - 1)^2}{2\tau_i}$$

The other dual weights are

$$\omega_{1i}^* = \frac{\tau_i}{2}, \quad \omega_{2i}^* = \tau_i - 1, \quad \omega_{4i} = \frac{1}{2}, \quad \omega_{5i}^* = \frac{\tau_i - 1}{2\tau_i} \quad \text{where } \tau_i > 1$$

Substituting the dual weights into the dual function, we get dw_i^*. Following Duffin et al. (1967) we get the optimum objective value as $TC_{li}^* = dw_i^*$. The decision variables can be obtain from the following relations:

$$\frac{h_i c_{0i} d_{2i}^{-\tau_i} M_i^{-\tau_i \eta_i} Q_i}{2\omega_{1i}^*} = dw_i^*, \quad \frac{c_{3i} d_{2i} M_i^{\eta_i}}{\omega_{4i}^* Q_i} = dw_i^*, \quad \frac{c_{2i} S_i^2}{\omega_{5i}^* Q_i} = dw_i^*$$

Solving the above relations, we get

$$M_i^* = \left(\frac{h_i c_{0i} c_{3i}}{2 d_{2i}^{\tau_i - 1} \omega_{1i}^* \omega_{4i}^* (dw_i^*)^2} \right)^{\frac{1}{\eta_i(\tau_i - 1)}}$$

$$Q_i^* = \frac{c_{3i} d_{2i}}{\omega_{4i}^* dw_i^*} \left(\frac{h_i c_{0i} c_{3i}}{2 d_{2i}^{\tau_i - 1} \omega_{1i}^* \omega_{4i}^* (dw_i^*)^2} \right)^{\frac{1}{\tau_i - 1}}$$

$$S_i^* = \left(\frac{2\omega_{5i}^* dw_i^*}{c_{2i}} \frac{c_{3i} d_{2i}}{\omega_{4i}^* dw_i^*} \left(\frac{h_i c_{0i} c_{3i}}{2 d_{2i}^{\tau_i - 1} \omega_{1i}^* \omega_{4i}^* (dw_i^*)^2} \right)^{\frac{1}{\tau_i - 1}} \right)^{0.5}$$

The ideal objective value TC_{01} is defined as

$$TC_{01} = \sum_{i=1}^{n} TC_{1i}^{*}$$

In a similar way, we can find the optimal value of the objective function $TC_{2i}(M)$.

$$\text{Minimize} \quad TC_{2i}(M) = d_{2i} M_i^{\eta_i+1} + d_{2i}^{1-\tau_i} c'_{0i} M_i^{\eta_i(1-\tau_i)}$$

subject to $M > 0$.

The above problem is a primal problem with DD = 0.

The corresponding dual function is

$$\text{Maximize} \quad dw''_i = \left(\frac{d_{2i}}{\omega_{6i}}\right)^{\omega_{6i}} \left(\frac{d_{2i}^{1-\tau_i} c'_{0i}}{\omega_{7i}}\right)^{\omega_{7i}}$$

subject to the normality and orthogonal conditions

$$\omega_{6i} + \omega_{7i} = 1 \quad (\eta_i + 1)\omega_{6i} + \eta_i(1-\tau_i)\omega_{7i} = 0$$

where $\omega_{6i},\ \omega_{7i} > 0$.

Solving the dual weights, we get

$$\overset{*}{\omega}_{6i} = \frac{\eta_i(\tau_i - 1)}{\tau_i\eta_i + 1},\ \overset{*}{\omega}_{7i} = \frac{\eta_i + 1}{\tau_i\eta_i + 1}, \qquad \tau_i > 1.$$

Substituting the dual weights into the dual function, we get $dw_i''^{*}$. Following Duffin et al. (1967), the optimal objective value is $TC_{2i}^{*} = dw_i''^{*}$.

The optimal values of the decision variable are obtained from

$$\frac{d_{2i} M_i^{\eta_i+1}}{\omega_{6i}} = dw_i''^{*}$$

Solving the relation, we get

$$M_i''^{*} = \left(\frac{\omega_{6i}^{*} \, d w_i''^{*}}{d_{2i}} \right)^{\frac{1}{\eta i+1}} .$$

The ideal objective TC_{02} is defined as $TC_{02} = \sum_{i=1}^{n} TC_{2i}^{*}$.

FUZZY GEOMETRIC PROGRAMMING WITH NUMERICAL EXAMPLES

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Abstract: Geometric programming (GP) has the high potential to be applied to a wide range of problems. This chapter summarizes the fundamentals of fuzzy GP and presents many application examples. Some variants of the gravel box problem are presented to solve it by fuzzy GP.

Key words: Fuzzy GP, gravel box problem, posynomial, signomial

1. INTRODUCTION

1.1 Geometric Programming

Geometric programming (GP) can be considered to be an innovative modus operandi to solve a nonlinear problem in comparison with other nonlinear techniques. It was originally developed to design engineering problems. It has become a very popular technique since its inception in solving non-linear problems. The concept of geometric programming (GP) was introduced by Duffin et al. (1967) in their famous book *Geometric Programming—Theory and Application*. This publication is a landmark in the development of GP. It studied all the theoretical developments up to date providing important examples to illustrate the technique. In addition to elegant proofs, it provided several constructive transformations and approximation for expressing optimization problems in suitable form in order to solve by GP.

C. Kahraman (ed.), Fuzzy Multi-Criteria Decision Making.

The study of GP by Duffin et al. (1967) deals with the problem involving only a positive coefficient for the component cost terms. However, many real-world problems comprise of positive as well as negative coefficients for the cost terms. Passy and Wilde (1967) made a significant methodological development of GP to deal with this type of problem. They extended the concept of the GP technique to generalized polynomials free from a restrictive environment. Now GP is capable of dealing with any problems involving signomials in both objective and constraint functions. It is important to note that any nonlinear algebraic problem can be transformed into an equivalent posynomial/signomial. For a detailed discussion, one may consult with the book *Applied Geometric Programming* written by Beightler and Phillips (1976).

The advantages of GP are as follows:

- This technique provides us with a systematic approach for solving a class of nonlinear optimization problems by finding the optimal value of the objective function and then the optimal values of the design variables are derived.
- This method often reduces a complex nonlinear optimization problem to a set of simultaneous equations.
- This approach is more amenable to the digital computers.
- This method allows an easy sensitivity analysis, which can be performed in the optimal solution.

GP inherits some drawbacks. The main disadvantages of GP lie in the fact that it requires the objective functions and constraints in the form of posynomials/signomials.

Note. Someone guesses that the name GP comes from the many geometrical problems. There is a difference between GP and geometric optimization (GOP). GP is an optimization problem based on the arithmetic- geometric mean inequality (A.M. ≥ G.M.). However, GOP is an optimization problem involving geometry.

GP is an optimization problem of the form

$$\text{Minimize } g_0\ (t) \tag{1}$$

subject to

$$g_j(t) \le 1, \qquad j = 1, 2, \ldots, m$$

$$h_k(t) = 1, \qquad k = 1, 2, \ldots, p$$

$t_i > 0, \qquad i = 1, 2, \ldots, n$

where $g_j(t)$ ($j = 0, 1, 2, \ldots, m$) are posynomial or signomial functions, $h_k(t)$ ($k = 1, 2, \ldots, p$) are monomials and t is the decision variable vector of n components t_i ($i = 1, 2, .., n$).

The problem (1) may be written as:

$$\text{Minimize } g_0(t) \tag{2}$$
$$\text{subject to } g'_j(t) \le 1 \qquad j = 1, 2, \ldots, m$$

$t > 0$, [since $g_j(t) \le 1, h_j(t) = 1 \Rightarrow g'_j(t) \le 1$ where $g'_j(t)$ ($= g_j(t)/h_k(t)$) be a posynomial ($j = 1, 2, .., m$; $k = 1, 2, \ldots, p$)].

2. POSYNOMIAL GEOMETRIC PROGRAMMING PROBLEM

2.1 Primal Problem

$$\text{Minimize } g_0(t) \tag{3}$$

subject to

$$g_j(t) \le 1, \ (j = 1, 2, \ldots, m)$$

and $t_i > 0$, ($i = 1, 2, \ldots, n$)

where $g_j(t) = \sum_{k=1}^{N_j} c_{jk} \prod_{i=1}^{n} t_i^{\alpha_{jki}}$

where c_{jk}(> 0) and α_{jki} ($i = 1, 2, \ldots, n$; $k = 1, 2, \ldots, N_j$; $j = 0, 1, 2, \ldots, m$) are real numbers.

$t \equiv (t_1, t_2, \ldots, t_n)^T$.

It is a constrained posynomial primal geometric problem (PGP). The number of inequality constraints in the problem (3) is m. The number of terms in each posynomial constraint function varies, and it is denoted by N_j for each $j = 0, 1, 2, \ldots, m$.

The degree of difficulty (DD) of a GP is defined as number of terms in a PGP – (number of variables in PGP + 1).

2.2 Dual Problem:

The dual programming of (3) is as follows:

$$\text{Maximize } d(w) = \prod_{j=0}^{m} \prod_{k=1}^{N_j} \left(\frac{c_{jk} w_{j0}}{w_{jk}} \right)^{w_{jk}} \tag{4}$$

subject to

$$\sum_{k=1}^{N0} w_{0k} = 1 \quad \text{(normality condition)}$$

$$\sum_{j=0}^{m} \sum_{k=1}^{Nj} \alpha_{jki} w_{jk} = 0, \ (i = 1, 2, ..., n) \quad \text{(orthogonality condition)}$$

$$w_{j0} = \sum_{k=1}^{Nj} w_{jk} \geq 0, \ w_{jk} \geq 0, \quad (i = 1, 2, ..., n; \quad k = 1, 2, \ldots, N_j), \ w_{00} = 1.$$

There are $n+1$ independent dual constraint equalities and $N = \sum_{j=1}^{m} N_j$ independent dual variables for each term of the primal problem. In this case $DD = N-(n+1)$.

3. SIGNOMIAL GEOMETRIC PROGRAMMING PROBLEM

3.1 Primal Problem

$$\text{Minimize } g_0(t) \tag{5}$$

subject to

$$g_j(t) \leq \delta_j, \quad (j = 1, 2, \ldots, m)$$

and $t_i > 0$, $(i = 1, 2, \ldots, n)$

where $g_j(\mathrm{t}) = \sum_{k=1}^{N_j} \delta_{jk}\ c_{jk} \prod_{i=1}^{n} t_i^{\alpha_{jki}}$ $(j = 0, 1, 2, \ldots, m)$

$\delta_j = \pm 1$ $(j = 2, \ldots, m)$, $\delta_{jk} = \pm 1$ $(j = 0, 1, 2, \ldots, m;\quad k = 1, 2, \ldots, N_j)$, $t \equiv (t_1, t_2, \ldots, t_n)^T$.

3.2 Dual Problem

The dual problem of (5) is as follows:

$$\text{Maximize } d(w) = \delta_0 \left[\prod_{j=0}^{m} \prod_{k=1}^{N_j} \left(\frac{c_{jk} w_{j0}}{w_{jk}} \right)^{\alpha_{jk} \quad w_{jk}} \right]^{\delta_0} \tag{6}$$

subject to

$$\sum_{k=1}^{N_0} \delta_{0k}\ w_{0k} = \delta_0 \qquad \text{(normality condition)}$$

$$\sum_{j=0}^{m} \sum_{k=1}^{N_j} \delta_{jk}\ \alpha_{jki} w_{jk} = 0 \qquad (i = 1, 2, ..., n) \quad \text{(orthogonality condition)}$$

where $\delta_j = \pm 1$ (j = 2, …, m), $\delta_{jk} = \pm 1$ $(j = 1, 2, \ldots, m; k = 1, 2, \ldots, N_j)$, and $w_{00} = 1$

$\delta_0 = +1, -1$ and non-negativity conditions, $w_{j0} \equiv \delta_j \sum_{k=1}^{N_j} \delta_{jk} w_{jk} \geq 0$, $\delta_{jk} \geq 0$, $(j = 1, 2, \ldots, m; k = 1, 2, \ldots, N_j)$ and $w_{00} = 1$.

4. FUZZY GEOMETRIC PROGRAMMING (FGP)

$$\widetilde{\text{Minimize}}\ g_0(t) \tag{7}$$

subject to $g_j(t) \underset{\sim}{\leq} b_j$ $(j = 1,2, \ldots, m)$

$t \geq 0$

Here, the symbol "$\widetilde{Minimize}$" denotes a relaxed or fuzzy version of "Minimize." Similarly, the symbol "$\lesssim$" denotes a fuzzy version of "≤." These fuzzy requirements may be quantified by eliciting membership functions $\mu_j(g_j(t))$ (j = 0, 1, 2,…, m) from the decision maker for all functions $g_j(t)$ (j = 0, 1, 2,…, m). By taking account of the rate of increased membership satisfaction, the decision maker must determine the subjective membership function $\mu_j(g_j(t))$. It is, in general, a strictly monotone decreasing linear or non linear function $u_j(g_j(t))$ with respect to $g_j(t)$ (j = 0, 1, 2,…, m). Here for simplicity, linear membership functions are considered. The linear membership functions can be represented as follows:

$$\mu_j(g_j(t)) = \begin{cases} 1, & \text{if } g_j(t) \le g_j^0 \\ \dfrac{g_j' - g_j(t)}{g_j' - g_j^0} & \text{if } g_j^0 \le g_j(t) \le g_j' \\ 0 & \text{if } g_j(t) \ge g_j' \end{cases}$$

for j = 0, 1, 2,…, m.

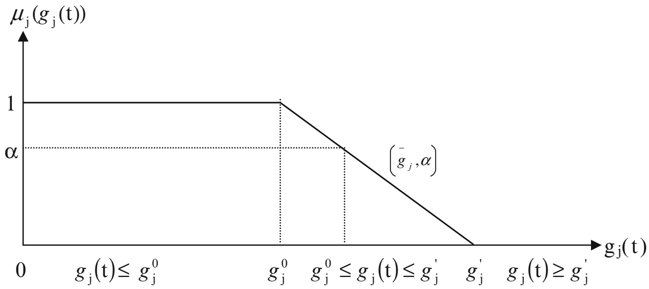

Figure 1. Membership function

As shown in Figure 1,

$g_j^0 \equiv$ the value of $g_j(t)$ such that the grade of membership function $\mu_j(g_j(t))$ is 1.

$g_j^{'}$ ≡ the value of $g_j(t)$ such that the grade of membership function $\mu_j\left(g_j(t)\right)$ is 0.

$\bar{g}_j$ ≡ the intermediate value of $g_j(t)$ between g_j^0 and $g_j^{'}$ (i.e., $\bar{g}_j \in (g_j^0, g_j^{'})$) such that the grade of membership function α ∈ (0,1).

The problem (7) reduces to FGP when $g_0(t)$ and $g_j(t)$ are signomial and posynomial functions.

Based on fuzzy decision making of Bellman and Zadeh (1972), we may write

(i) $\mu_D\left(t^*\right) = max\left(min\,\mu_j\left(g_j(t)\right)\right)$ (max−min operator) (8)

subject to

$$\mu_j\left(g_j(t)\right) = \frac{g_j^{'} - g_j(t)}{g_j^{'} - g_j^0} \quad (j = 0, 1, 2, \ldots, m)$$

$t > 0$

(ii) $\mu_D\left(t^*\right) = \max\left(\sum_{j=0}^{m} \lambda_j \mu_j\left(g_j(t)\right)\right)$ (max−additive operator) (9)

subject to $\mu_j\left(g_j(t)\right) = \dfrac{g_j^{'} - g_j(t)}{g_j^{'} - g_j^0}$ $(j = 0, 1, 2,\ldots, m)$

$t > 0$

(iii) $\mu_D\left(t^*\right) = \max\left(\prod_{j=0}^{m} \left(\mu_j\left(g_j(t)\right)\right)^{\lambda_j}\right)$ (max−product operator) (10)

subject to

$$\mu_j\left(g_j(t)\right) = \frac{g_j^{'} - g_j(t)}{g_j^{'} - g_j^0} \quad (j = 0, 1, 2, \ldots, m)$$

$t > 0.$

Here, for λ_j (j = 0, 1, 2, ..., m) are numerical weights considered by a decision making unit. For normalized weights

$$\sum_{j=0}^{m} \lambda_j = 1 \text{ and } \lambda_j \in [0,1]$$

For equal importance of objective and constraint goals, $\lambda_j = 1$.

In contrast to GP, FGP in general has not been widely circulated in the literature. In 1990, Verma studied a new concept to use the GP technique for multi-objective fuzzy decision-making problems. He projected the very importance on the product operator, which reduces the DD with a considerable amount. Biswal (1992) applied the fuzzy programming technique to solve a multi-objective GP problem as a vector minimization problem. A vector maximization problem can be transformed into a vector minimization problem. Cao (1993, 1994) discussed the properties of a kind of posynomial GP with an L-R fuzzy coefficient in objectives and constraints. In the sequel, Cao (2002) published an important book on FGP, which was the most recent book until now.

If $g_j(t)$ (j = 0,1,2,…,m) be posynomial function as

$g_j(t) = \sum_{k=1}^{N_j} c_{jk} \prod_{i=1}^{n} t_i^{\alpha_{jki}}$ (c_{jk}(> 0) and α_{jki} (i = 1, 2, ..., n; k = 1, 2,..,N_j; j = 0, 1, 2,…, m)) then

i) max−min operator (8) reduces to

$$\text{Maximize} \left(\lambda_j \frac{g_j^{'} - g_j(t)}{g_j^{'} - g_j^0} \right)$$

subject to

$$\lambda_r \frac{g_r^{'} - g_r(t)}{g_r^{'} - g_r^0} \geq \lambda_j \frac{g_j^{'} - g_j(t)}{g_j^{'} - g_j^0}, \quad (r = 0, 1, 2, \ldots, m \text{ and } r \neq j)$$

$t > 0$.

So $V_M^*(t^*) = \lambda_j \dfrac{g_j'}{g_j' - g_j^0} - \; V^*(t^*)$

where t^* is obtained by solving the following signomial GP:

Minimize $V(t) = \dfrac{\lambda_j}{g_j' - g_j^0} - \sum_{k=1}^{N_j} c_{jk} \prod_{i=1}^{n} t_i^{\alpha_{jki}}$ (12)

subject to

$$\frac{\dfrac{\lambda_r}{g_r' - g_r^0}\sum_{k=1}^{N_j} c_{rk}\prod_{i=1}^{n} t_i^{\alpha_{rki}} - \dfrac{\lambda_j}{g_j' - g_j^0}\sum_{k=1}^{N_j} c_{jk}\prod_{i=1}^{n} t_i^{a_{jki}}}{\dfrac{\lambda_r g_r'}{g_r' - g_r^0} - \dfrac{\lambda_j g_j'}{g_j' - g_j^0}}$$

($r = 0, 1, 2, \ldots, m$ and $r \neq j$)

$t > 0$

ii) max−additive operator (9) reduces to

Maximize $V_A(t) = \sum_{j=0}^{m} \lambda_j \dfrac{g_j' - \sum_{k=1}^{N_j} c_{jk} \prod_{i=1}^{n} t_i^{a_{jki}}}{g_j' - g_j^0}$ (13)

subject to

$t > 0$

So the optimal decision variable t^* with the optimal objective value $V^*(t^*)$ can be obtained by $V^*(t^*) = \sum_{j=0}^{m} \dfrac{\lambda_j g_j'}{g_j' - g_j^0} - U^*(t^*)$ where t^* is the optimal decision variable of the unconstrained geometric programming problem

$$\text{Minimize } U(t) = \sum_{j=0}^{m} \frac{\lambda_j}{g_j^{'} - g_j^{0}} \sum_{k=1}^{N_j} C_{jk} \prod_{i=1}^{n} t_i^{a_{jki}} \tag{14}$$

subject to

$t > 0.$

iii) Similarly, Eq. (10) can be solved by GP based on a suitable transformation.

4.1 Numerical Example 1

$$\text{Minimize } \; Z(x) = x_1^{-1} x_2^{-2}$$

$$\text{[Here objective goal } Z(x) \underset{\sim}{\le} 6.94 \text{ with tolerance } 0.19 \text{]} \tag{15}$$

$$Y_1(x) = 2x_1^{-2} x_2^{-3} \tilde{\le} \; 57.87 \text{ (with tolerance 2.88)}$$

$$Y_2(x) = x_1 + x_2 \le 1, \qquad x_1, x_2 > 0$$

Here, linear membership functions for the fuzzy objective and constraint goals are

$$\mu_1(x) = \begin{cases} 1, & \text{if } Z_1(x) \le 6.75 \\ \dfrac{6.94 - Z_1(x)}{0.19} & \text{if } 6.75 \le Z_1(x) \le 6.94 \\ 0 & \text{if } Z_1(x) \ge 6.94 \end{cases}$$

$$\mu_2(x) = \begin{cases} 1, & \text{if } Y_1(x) \le 57.87 \\ \dfrac{60.75 - Y_1(x)}{2.88} & \text{if } 57.85 \le Y_1(x) \le 60.75 \\ 0 & \text{if } Y_1(x) \ge 60.75 \end{cases}$$

i) Based on max–min operator (8), FGP (15) reduces to

$$\text{Maximize } V(x_{1,}x_2) = \frac{6.94 - x_1^{-1}x_2^{-2}}{0.19} \quad (16)$$

$$\text{subject to } \frac{60.75 - x_1^{-1}x_2^{-2}}{2.88} \geq \frac{6.94 - x_1^{-1}x_2^{-2}}{0.19}$$

$x_1 + x_2 \leq 1, \quad x_1 > 0, \quad x_2 > 0$

So Eq. (16) reduces to

$$\text{Maximize } V_M(x_{1,}x_2) = 36.37 - \text{V}(x_{1,}x_2) \quad (17)$$

subject to

$0.341\, x_1^{-1}x_2^{-2} - 0.045\ x_1^{-2}x_2^{-3} \geq 1$, $x_1 > 0$, $x_2 > 0$

where $V(x_1, x_2) = 5.26\, x_1^{-1}x_2^{-2}$

To solve Eq. (17), we are to solve the following crisp GP:

Minimize $V(x_1, x_2) = 5.26\, \text{x}_1^{-1}\text{x}_2^{-2}$.

subject to

$0.45\ x_1^{-2}x_2^{-3} - 0.341\ x_1^{-1}x_2^{-2} \leq -1$, $x_1 + x_2 \leq 1$, $x_1 > 0$, $x_2 > 0$.

For this problem DD = 5 − (2 + 1) = 2.

The dual problem (DP) of this GP is

Maximize $d(w_{01}, w_{11}, w_{12}, w_{21}, w_{22}, \sigma_0)=$

$$\sigma_0 \begin{Bmatrix} \left(\frac{5.26}{w_{01}}\right)^{w_{01}} \left(\frac{0.045}{w_{11}}\right)^{w_{11}} \left(\frac{0.341}{w_{12}}\right)^{-w_{12}} \left(\frac{1}{w_{21}}\right)^{w_{21}} \\ \left(\frac{1}{w_{22}}\right)^{w_{22}} \left(-w_{11}+w_{12}\right)^{w_{11}-w_{12}} \left(w_{21}+w_{22}\right)^{w_{21}+w_{22}} \end{Bmatrix}^{\sigma_0}$$

subject to

$$\sigma_0 w_{01} = 1$$

$$-w_{01} - 2w_{11} + w_{12} + w_{21} = 0, \quad -2w_{01} - 3w_{11} + 2w_{12} + w_{22} = 0$$

Considering $\sigma_0 = 1$, we have

$$w_{01} = 1, w_{21} = 2w_{11} - w_{12} + 1, \; w_{22} = 3w_{11} - 2w_{12} + 2$$

Here $\mu_1 = \left(\sigma_{11} w_{11} + \sigma_{12} w_{12}\right) \;\; \sigma_1 = \;\; -w_{11} + w_{12} \;\geq\; 0$.

$$\text{So, } \max d\left(w_{11}, w_{12}\right) = 5.26 \left(\frac{0.045}{w_{11}}\right)^{w_{11}} \left(\frac{0.341}{w_{10}}\right)^{-w_{12}} \left(\frac{1}{2w_{11} - w_{12} + 1}\right)^{2w_{11}-w_{12}+1} \times \left(\frac{1}{3w_{11} - 2w_{12} + 2}\right)^{3w_{11}-2w_{12}+2}$$

$$\left(w_{12} - w_{11}\right)^{w_{11}-w_{12}} \left(5w_{11} - 3w_{12} + 3\right)^{5w_{11}-3w_{12}+3}$$

For optimality of $d\left(w_{11}, w_{12}\right)$, we have

$$\frac{\partial d\left(w_{11}, w_{12}\right)}{\partial w_{11}} = 0$$

and

$$\frac{\partial d\left(w_{11}, w_{12}\right)}{\partial w_{12}} = 0 \, .$$

That is,

$$0.045\left(w_{12} - w_{11}\right)\ \left(5w_{11} - 3w_{12} + 3\right)^5 = w_{11}\left(2w_{11} - w_{12} + 1\right)^2\left(3w_{11} - 2w_{12} + 2\right)^3$$

and

$$w_{12}(2w_{11} - w_{12} + 1)(3w_{11} - 2w_{12} + 2) = 0.341(w_{12} - w_{11})(5w_{11} - 3w_{12} + 3)^3 \, .$$

the optimal solution is $d(w) = 35.75646422$, $w_{01} = 1$, $w_{11} = 0.2901869$, $w_{12} = 0.5103887$, $w_{21} = 1.0699851$, $w_{22} = 1.8497833$, $\mu_1 = 0.2202018$.

So, the optimal solution of Eq. (13) is x_1 = 0.36631095, x_2 = 0.633699788, and $Z(x)$ = 6.79798859.

ii) Based on the max−additive operator (9), FGP (15) reduces to

Maximize

$$V_A(x_1,x_2) = \left(\frac{6.94 - x_1^{-1}x_2^{-2}}{0.19} + \frac{60.75 - x_1^{-1}x_2^{-2}}{2.88}\right) = \left\{57.62 - V(x_1,x_2)\right\} \qquad (18)$$

subject to

$x_1 + x_2 \leq 1$, $x_1 > 0$, $x_2 > 0$

where $V(x_1, x_2) = 5.263x_1^{-1}x_2^{-2} + 0.694x_1^{-2}x_2^{-3}$.

To solve Eq. (18), we are to solve the following crisp GP:

Minimize $V(\mathrm{x}_1, \mathrm{x}_2) = 5.263x_1^{-1}x_2^{-2} + 0.694x_1^{-2}x_2^{-3}$

subject to

$x_1 + x_2 \leq 1$, $x_1 > 0$, $x_2 > 0$.

For this problem, DD = 4−(2 + 1) = 1

The DP of this GP is

$$\text{Maximize} \quad d(w_{01}, w_{02}, w_{11}, w_{12})$$

$$= \left(\frac{5.263}{w_{01}}\right)^{w_{01}} \left(\frac{0.694}{w_{02}}\right)^{w_{02}} \left(\frac{1}{w_{11}}\right)^{w_{11}} \left(\frac{1}{w_{12}}\right)^{w_{12}} \left(w_{11} + w_{12}\right)^{w_{11}+w_{12}}$$

subject to

$$w_{01} + w_{02} = 1, \qquad -w_{01} - 2w_{02} + w_{11} = 0$$

$$-2w_{01} - 3w_{02} + w_{12} = 0$$

So, $w_{02} = 1 - w_{01}$, $w_{11} = 2 - w_{01}$, $w_{12} = 3 - w_{01}$

$$\therefore \quad \text{Maximize} \quad d\ (w_{01}) = \left(\frac{5.263}{w_{01}}\right)^{w_{01}} \left(\frac{0.694}{1-w_{01}}\right)^{1-w_{01}} \left(\frac{1}{2-w_{01}}\right)^{2-w_{01}}$$

$$\left(\frac{1}{3-w_{01}}\right)^{3-w_{01}} \left(5-2w_{01}\right)^{5-2w_{01}}$$

subject to $0 < w_{01} < 1$

For optimality of $d(w_{01}, w_{02}, w_{11}, w_{12})$, we have

$$\frac{d}{dw_{01}}\left(d(w_{01})\right) = 0$$

That is, $5.263(1-w_{01})\ (2-w_{01})\ (3-w_{01}) = 0.694 w_{01}(5-2w_{01})^2$.

The optimal solution is $d(w_{01}) = 56.10412298$

$w_{01} = 0.6375822$, $w_{02} = 0.3624178$, $w_{11} = 1.3624178$, and $w_{12} = 2.3624178$

So, the optimal solution of Eq. (15) is $x_1 = 0.364517711$, $x_2 = 0.635681102$, and $Z(x) = 6.788952396$

iii) Based on the max−product operator (10), FGP (15) reduces to

Maximize

$$V_p(x_1,x_2)=\left(\frac{6.94-x_1^{-1}x_2^{-2}}{0.19}\right)\times\left(\frac{60.75-x_1^{-1}x_2^{-2}}{2.88}\right)=770.479-V(x_1,x_2) \quad (19)$$

subject to

$x_1+x_2 \le 1, \quad x_1 > 0, \quad x_2 > 0$

where $V(x_1,x_2)=111.018x_1^{-1}x_2^{-2}+25.349x_1^{-2}x_2^{-3} - 3.653x_1^{-3}x_2^{-5}$.

To solve Eq. (19), we are to solve the following crisp GP:

Minimize $V(x_1,x_2)=111.018\ x_1^{-1}x_2^{-2}+25.3492x_1^{-2}x_2^{-3} - 3.653x_1^{-3}x_2^{-5}$

subject to

$x_1+x_2 \le 1, \quad x_1 > 0, \quad x_2 > 0$

For this case DD = 5−(2 + 1) = 2.

The DP of this GP is

Maximize $\quad d(w_{01},w_{02},w_{11},w_{12})$

$$=\left(\frac{111.018}{w_{01}}\right)^{w_{01}}\left(\frac{25.349}{w_{02}}\right)^{w_{02}}\left(\frac{3.653}{w_{03}}\right)^{w_{03}}\left(\frac{1}{w_{11}}\right)^{w_{11}}$$

$$\left(\frac{1}{w_{12}}\right)^{w_{12}}\left(w_{11}+w_{12}\right)^{w_{11}+w_{12}}$$

subject to

$$w_{01}+w_{02}-w_{03}=1$$

$$-w_{01}-2w_{02}+3w_{03}+w_{11}=0$$

$-2w_{01} - 3w_{02} + 5w_{03} + w_{12} = 0$

So, $w_{03} = w_{01} + w_{02} - 1$

$w_{11} = 3 - 2w_{01} - w_{02}$

$w_{12} = 5 - 3w_{01} - 2w_{02}$

For the optimality of $d(w_{01}, w_{02}, w_{11}, w_{12})$, we have

$$\frac{\partial}{\partial w_{01}} d(w_{01}, w_{02}, w_{11}, w_{12}) = 0, \quad \frac{\partial}{\partial w_{02}} d(w_{01}, w_{02}, w_{11}, w_{12}) = 0.$$

That is $111.018\,(3 - 2w_{01} - w_{02})^2\,(5 - 3w_{01} - 2w_{02})^3 (w_{01} + w_{02}^{-1}) = 3.653 w_{01}\,(8 - 5w_{01} - 3w_{02})^5$

and

$25.349\,(3 - 2w_{01} - w_{02})\,(5 - 3w_{01} - 2w_{02})^2 (w_{01} + w_{02} - 1)$

$= 3.653 w_{02}\,(8 - 5w_{01} - 3w_{02})^3$. The optimal solution is $d(w) = 769.8551092$,

$w_{01} = 0.9774833$, $w_{02} = 0.9249442$,
$w_{03} = 0.9024275$, $w_{11} = 0.1200892$, and $w_{12} = 0.2176617$

So, the optimal solution of Eq. (15) is $x_1 = 0.394682105$, $x_2 = 0.611383637$, and $Z(x) = 6.778364884$.

5. APPLICATION

5.1 Gravel Box Problem

Problem 1a.
A total of 80 cubic-meters of gravel is to be ferried across a river on a barge. A box (with an open top) is to be built for this purpose. After the entire gravel has been ferried, the box is to be discarded. The transport cost per round trip of barge of box is *Rs* 1; the cost of materials of the sides and bottom of the box are *Rs* 10/m^2 and *Rs* 80/m^2 and the ends of box are *Rs* 20/m^2. Find the dimension of the box that is to be built for this purpose and the total optimal cost (see Figure 2).

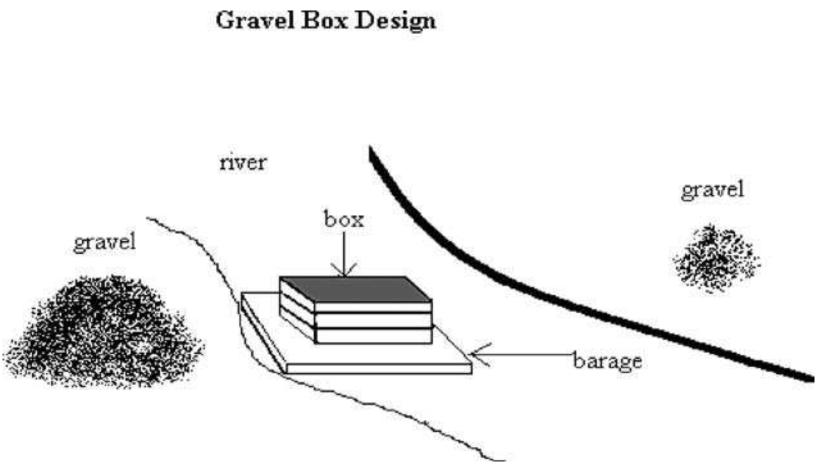

Figure 2. Gravel box problem

Let us assume the gravel box has

length = $t_1 m$, width $= t_2 m$, height $= t_3 m$

$\therefore$ The area of the ends of the gravel box $= t_2 t_3 m^2$

The area of the sides of the gravel box $= t_1 t_3 m^2$

The area of the bottom of the gravel box $= t_1 t_2 \ m^2$

$\therefore$ The volume of the gravel box $= t_1 t_2 t_3 m^3$

Cost function

Transport cost: $(Rs\,1/\text{trip})\dfrac{80\text{m}^3}{t_1 t_2 t_3 \text{m}^3/\text{trip}} = Rs.\ 80 t_1^{-1} t_2^{-1} t_3^{-1}$,

Material cost: Ends of box: $2\left(Rs\,20/\text{m}^2\right) t_2 t_3 \text{m}^2 = Rs.\ 40 t_2 t_3$

Sides of box : $2\left(Rs\,10/\text{m}^2\right) t_1 t_3 \text{m}^2 = Rs.\ 20 t_1 t_3$

Bottom of box: $\left(Rs\ \ 80/\text{m}^2\right) t_1 t_2 \text{m}^2 = Rs.80\ \ t_1 t_2$

The total cost (Rupees):

$$g(t) = 80 t_1^{-1} t_2^{-1} t_3^{-1} + 40 t_2 t_3 + 20 t_1 t_3 + 80 t_1 t_2$$

It is a posynomial function.

As stated, this problem can be formulated as an unconstrained GP problem

$$\text{Minimize} \quad g(t) = 80\ \ t_1^{-1} t_2^{-1} t_3^{-1} + 40 t_2 t_3 + 20 t_1 t_3 + 80 t_1 t_2$$

subject to $t_1, t_2, t_3 > 0$

The optimal dimensions of the box are $t_1^* = 1$m, $t_2^* = 1/$ m, and $t_3^* = 2$m, and the minimum total cost of this problem is Rs 200.

Problem 1b.

We now consider the following variant of the above problem (a similar discussion take place in Duffin et al., 1967 in their book). It is required that the sides and bottom of the box should be made from scrap material, but only 4 m^2 of this scrap material are available.

This variation of the problem leads us to the following constrained posynomial GP problem:

$$\begin{cases} \text{Minimize} \quad g_0(t) = \dfrac{80}{t_1 t_2 t_3} + 40 t_2 t_3 \\ \text{subject to} \quad g_1(t) \equiv 2t_1 t_3 + t_1 t_2 \le 4 \\ \text{where} \quad t_1 > 0,\ t_2 > 0,\ t_3 > 0 \end{cases}$$

Solving this constrained GP problem, we have the minimum total cost Rs 95.24, and the optimal dimensions of the box are t_1^*=1.58m, t_2^*=1.25m, and t_3^*=0.63 m.

Problem 1c.

We now consider the fuzzy objective and constraint goal in Problem 1b. The fuzzy problem becomes

Find $t = (t_1, t_2, t_3)^T$ so as to satisfy

$$g_0(t) \lesssim 90 \text{ and } g_1(t) \lesssim 4, \quad t > 0$$

For treating the above fuzzy inequalities, we construct the following linear membership functions:

$$\mu_{g_1}(t_1,t_2,t_3) = \begin{cases} 0, & \text{if } g_1(t_1,t_2,t_3) > 6 \\ 1 - \dfrac{g_1(t_1,t_2,t_3) - 4}{2}, & \text{if } 4 < g_1(t_1,t_2,t_3) < 6 \\ 1, & \text{if } g_1(t_1,t_2,t_3) < 4 \end{cases}$$

$$\mu_{g_0}(t) = \begin{cases} 1, & \text{if } g_0(t) < 90 \\ \dfrac{98\text{-}g_0(t)}{8}, & \text{if } 90 \le g_0(t) \le 98 \\ 0, & \text{if } g_0(t) \ge 98 \end{cases}$$

$$\mu_{g_1}(t) = \begin{cases} 1, & \text{if } g_1(t) \le 4 \\ \dfrac{6\text{-}g_1(t)}{2}, & \text{if } 4 \le g_1(t) \le 6 \\ 0, & \text{if } g_1(t) \ge 6 \end{cases}$$

where 8 (= 98−90) and 2 (= 6−4) are subjectively chosen constants expressing the limit of the admissible violations of the inequalities.

It is assumed that the membership function $\mu_{g0}(t)$ should be 1 if the objective goal is well satisfied, and 0 if the objective goal is violated beyond its limit 8(= 98−90) and linear from 0 to 1.

Following the fuzzy decision on the max−additive operator (9), the said problem can be transformed into the following equivalent conventional nonlinear programming problem as

$$\text{Maximize } V(t) = \frac{98 - g_0(t)}{8} + \frac{6 - g_1(t)}{2}$$

subject to t > 0.

So the optimal decision variable t^* can be obtained by solving the following unconstrained GP problems:

$$\text{Minimize } U(t) = \frac{g_0(t)}{8} + \frac{g_1(t)}{2}$$

subject to t > 0.

$$\text{Minimize } U(t) = 10t_1^{-1}t_2^{-1}t_3^{-1} + 5t_2t_3 + t_1 t_3 + \frac{1}{2}t_1 t_2$$

subject to $t_1, t_2, t_3 > 0$.

Solving the unconstrained GP we have t_1^* = 1.58, t_2^*= 1.426883, and t_3^* = 0.7134413; the optimal objective goal is $g_0^*(t^*) = \frac{80}{t_1^* t_2^* t_3^*} + 40t_2^* t_3^*$ = 90.45766 and the constraint goal $g_1^*(t^*) = 2t_1^* t_3^* + t_1^* t_2^*$ = 4.50895.

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